The reverse mathematics of ω-automata

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Available at: leonardopacheco.xyz/slides/wakate2023.pdf

The Big Five

- Π^1_1-CA_0
- ATR_0
- ACA_0
- WKL_0
- RCA_0

The Big Five and Automata Theory

- Π^1_2-$\mathsf{CA}_0 \leftarrow$ complementation of tree-automata
- Π^1_1-CA_0
- ATR_0
- ACA_0
- $\mathsf{WKL}_0 \not\Leftrightarrow \Sigma^0_2$-$\mathsf{Ind} \leftarrow$ complementation of ω-automata
- RCA_0
- $\mathsf{RCA}^*_0 \leftarrow$ definitions

BÜCHI AUTOMATA

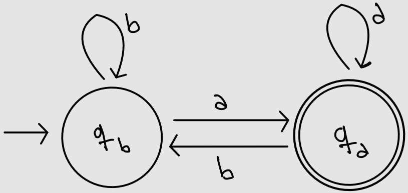
$$\mathcal{A} = \langle Q, \Sigma, q_i, \delta, F \rangle$$

- Q is a finite set of states
- Σ is a finite alphabet
- q_i is the initial state
- δ is the transition relation
- F is the set of accepting states

The automaton $\mathcal{A}$ accepts the infinite word $\alpha : \omega \to \Sigma$ iff there is a run ρ of $\mathcal{A}$ on α where $\mathrm{Inf}(\rho) \cap F \neq \emptyset$.

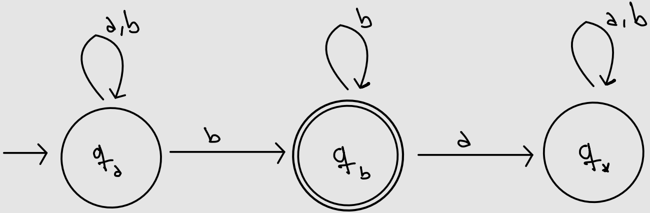
Büchi Automata — Example 1

$\mathcal{A}$ accepts $\alpha : \omega \to \{a, b\}$ iff a appears infinitely often in α.



BÜCHI AUTOMATA — EXAMPLE 2

$\mathcal{A}$ accepts $\alpha : \omega \to \{a, b\}$ iff a appears finitely often in α.



Reverse Mathematics

Theorem (Kołodziejczyk, Michalewski, Pradic, Skrzypczak[1])

The following are equivalent over RCA_0*:*

- Σ^0_2-Ind
- *complementation of Büchi automata*
- *Ramsey's Theorem for pairs restricted to additive colorings*
- *decidability of the MSO theory of* $\langle \mathbb{N}, \leq \rangle$

[1] Kołodziejczyk, Michalewski, Pradic, Skrzypczak, "The logical strength of Büchi's decidability theorem"

Muller Automata

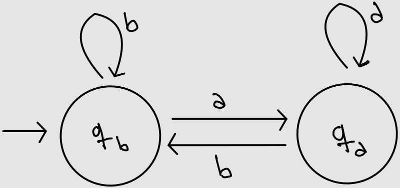
$$\mathcal{A} = \langle Q, \Sigma, q_i, \delta, T \rangle$$

- Q is a finite set of states
- Σ is a finite alphabet
- q_i is the initial state
- δ is the transition function
- F is the acceptance table

The automaton $\mathcal{A}$ accepts the infinite word $\alpha : \omega \to \Sigma$ iff there is a run ρ of $\mathcal{A}$ on α such that $\mathrm{Inf}(\rho) \in T$.

Muller Automata — Example 1

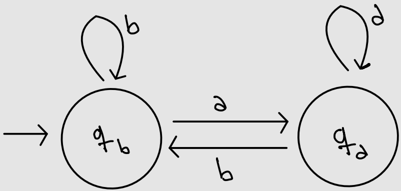
$\mathcal{A}$ accepts $\alpha : \omega \to \{a, b\}$ iff a appears infinitely often in α.



$T = \{\{q_a\}, \{q_a, q_b\}\}$

Muller Automata — Example 2

$\mathcal{A}$ accepts $\alpha : \omega \to \{a, b\}$ iff a appears finitely often in α.



$T = \{\{q_b\}\}$

Reverse Mathematics

Theorem (McNaughton)

Buchi automata and Muller automata are equivalent.

Theorem (Das[2])

"Certain formulations of McNaugton's theorem are not provable in RCA_0."

These formulations include all known proofs of the McNaughton theorem.

[2] Das, "On the logical complexity of cyclic arithmetic"

Tree Automata

$$\mathcal{A} = \langle Q, \Sigma, q_i, \delta, \Omega \rangle$$

- Q is a finite set of states
- Σ is a finite alphabet
- q_i is the initial state
- δ is the transition relation
- Ω is a parity function

The automaton $\mathcal{A}$ accepts the infinite tree $t : \{0, 1\}^* \to \Sigma$ iff there is a run of $\mathcal{A}$ on t where, on all paths, the infinitely often ocurring parity is even.

Tree Automata — Example 1

$\mathcal{A}$ accepts $t : \{0,1\}^* \to \Sigma$ iff a appears infinitely often in all branches of t.

$$q_0 \xrightarrow{a} q_a, q_a$$

$$q_0 \xrightarrow{b} q_b, q_b$$

$$\Omega(q_a) = 2$$

$$\Omega(q_b) = 1$$

Tree Automata — Example 2

$\mathcal{A}$ accepts $t : \{0,1\}^* \to \Sigma$ iff there is a branch of t where a appears finitely often.

$$q_0 \xrightarrow{a} q_a, q_*$$

$$q_0 \xrightarrow{a} q_*, q_a$$

$$q_0 \xrightarrow{b} q_b, q_*$$

$$q_0 \xrightarrow{b} q_*, q_b$$

$$q_* \xrightarrow{a,b} q_*, q_*$$

$$\Omega(q_a) = 1$$

$$\Omega(q_b) = 0$$

$$\Omega(q_*) = 0$$

Reverse Mathematics

Theorem (Kołodziejczyk, Michalewski[3])

The following are equivalent over Π^1_2-CA_0:

- *complementation of non-deterministic tree automata*
- *decidability of the Π^1_3 fragment of the MSO theory of the infinite binary tree*
- *the positional determinacy of parity games*
- *the determinacy of $\forall n.(\Sigma^0_2)$-Det Gale-Stewart games*
- *the reflection principle Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$*

[3] Kołodziejczyk, Michalewski, "How unprovable is Rabin's decidability theorem?"

MORE ON AUTOMATA

See Luke Ong's lecture notes "Automata, Logic and Games" for more on infinite automata. Available at:

https://www.cs.ox.ac.uk/people/luke.ong/personal/publications/ALG14-15.pdf

See Perrin and Pin's book "Infinite Words" for much more on infinite automata.

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- RCA^*_0 $\leftarrow$ definitions