On diamonds in constructive modal logic

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INTRODUCTION

Theorem (Das, Marin)

CK *and* IK *do not prove the same* $\Diamond$*-free formulas:*

- CK $\nvdash \neg\neg\Box\bot \to \Box\bot$, *and*
- IK $\vdash \neg\neg\Box\bot \to \Box\bot$

Theorem (P.)

CKB *and* IKB *prove the same formulas.*

Theorem (P.)

Over IEL, $\Diamond\varphi$ *is equivalent to* $\neg\neg\varphi$.

THE LOGIC CK

CK is the least set of formulas containing:

- intuitionistic tautologies;
- $K_{\Box} := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_{\Diamond} := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;

and closed under

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

THE LOGICS CKB, IK, AND IKB

Let

- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- $N := \neg\Diamond\bot$;
- $B_\Box := P \to \Box\Diamond P$; and
- $B_\Diamond := \Diamond\Box P \to P$.

Then:

- $\mathsf{CKB} := \mathsf{CK} + \{B_\Box, B_\Diamond\}$;
- $\mathsf{IK} := \mathsf{CK} + \{FS, DP, N\}$; and
- $\mathsf{IKB} := \mathsf{IK} + \{B_\Box, B_\Diamond\} = \mathsf{CKB} + \{FS, DP, N\}$.

CK-MODELS

A CK-model is a tuple $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $W^{\perp} \subseteq W$ is the set of *fallible worlds*;
- the *intuitionistic relation* $\preceq$ is a reflexive and transitive relation over W;
- the *modal relation* R is a relation over W;
- $V : \text{Prop} \to \mathcal{P}(W)$ is a *valuation function*.

We require:

- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- for all $P \in \text{Prop}$, $W^{\perp} \subseteq V(P)$;
- if $w \in W^{\perp}$ and either $w \preceq v$ or wRv, then $v \in W^{\perp}$.

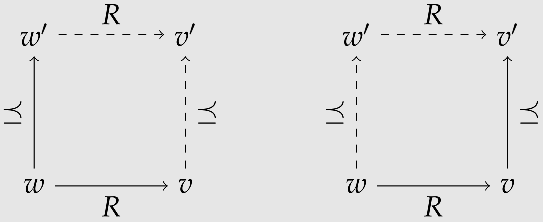
VALUATION

- $M, w \models P$ iff $w \in V(P)$;
- $M, w \models \bot$ iff $w \in W^{\bot}$;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \rightarrow \psi$ iff, for all $v \in W$, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$; and
- $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ then, there is u such that vRu and $M, u \models \varphi$.

IK-MODELS

An IK-model is a CK-model where:

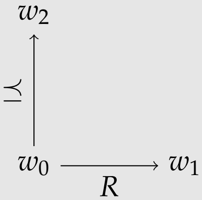
- $W^{\perp} = \emptyset$;
- R is forward and backward confluent:



An IKB-model is an IK-model where R is symmetric.

CK $\not\models \neg\neg\Box\bot \to \Box\bot$

Consider the model below:

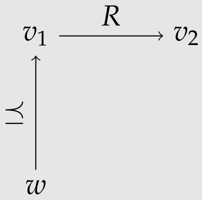


We have that $w_0 \models \neg\neg\Box\bot$ but $w_0 \not\models \Box\bot$.

$$(w \models \neg\neg\Box\bot \text{ iff } \forall v \succeq w \exists u \succeq v . u \models \Box\bot)$$

IK $\models \neg\neg\Box\bot \rightarrow \Box\bot$

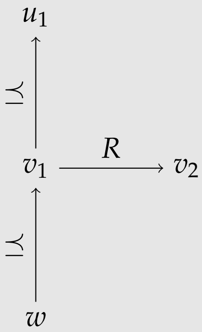
Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.



$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v. u \not\models \Box\bot)$

IK $\models \neg\neg\Box\bot \to \Box\bot$

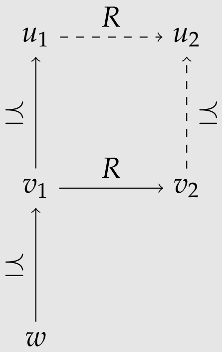
Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.



$$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v. u \not\models \Box\bot)$$

IK $\models \neg\neg\Box\bot \to \Box\bot$

Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.



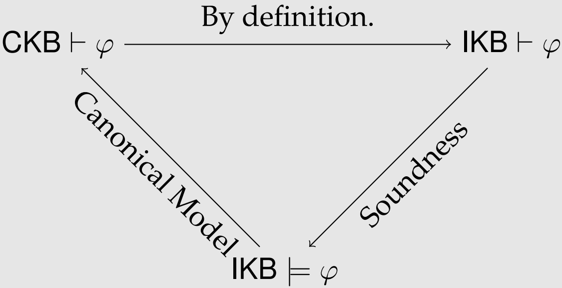
$$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v . u \not\models \Box\bot)$$

CKB AND IKB COINCIDE

Theorem

For all modal formula φ, the following are equivalent:

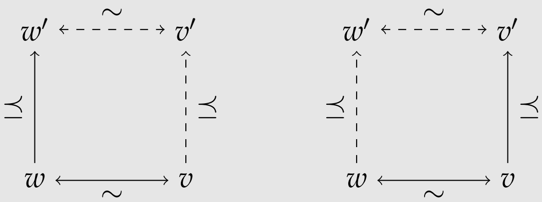
1. CKB $\vdash \varphi$;
2. IKB $\vdash \varphi$; *and*
3. IKB $\models \varphi$.



Symmetry Implies Confluences Coincide

Lemma

Let M be a CK*-model where the modal relation $\sim$ is symmetric. Then $\sim$ is forward confluent iff $\sim$ is backward confluent.*

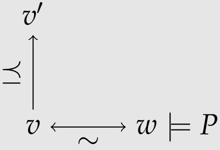


SYMMETRY IMPLIES CONFLUENCE IS NECESSARY

Lemma

There is a CK*-model* $M = \langle W, W^{\perp}, \preceq, \sim, V \rangle$ *and* $w \in W$ *such that:*

- $\sim$ *is a symmetric relation;*
- $B_{\Box} := P \rightarrow \Box\Diamond P$ *does not hold at* w.



Existing Results

Theorem (Arisaka, Das, Straßburger)

$\mathsf{CKB} \vdash DP$ *and* $\mathsf{CKB} \vdash N$.

Theorem (De Groot, Shillito, Clouston)

Let $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ *be a* CK*-model. Then:*

- *Suppose that, for all* $w, v \in W$*,* wRv*, and* $v \in W^{\perp}$ *implies* $w \in W^{\perp}$*. Then* $M \models N$*.*
- *Suppose that* R *is forward and backward confluent. Then* $M \models DP$ *and* $M \models FS$*.*

A Canonical Model for CKB

A (consistent) CKB-theory Γ is a set of formulas such that:

- Γ contains all the axioms of CKB and is closed under **MP**;
- if $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$;
- $\bot \notin \Gamma$.

Definition

The CKB-canonical model is $M_c := \langle W_c, W_c^{\perp}, \preceq_c, \sim_c, V_c \rangle$ where:

- $W_c := \{\Gamma \mid \Gamma \text{ is a CKB-theory}\}$;
- $W_c^{\perp} = \emptyset$;
- $\Gamma \preceq_c \Delta$ iff $\Gamma \subseteq \Delta$;
- $\Gamma \sim_c \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$ and $\Delta \subseteq \{\varphi \mid \Diamond\varphi \in \Gamma\}$;
- $\Gamma \in V_c(\varphi)$ iff $P \in \Gamma$.

Truth Lemma

Lemma

The CKB*-canonical model* M_c *is an* IKB*-model.*

The following lemma uses standard techniques:

Lemma

Let M_c *be the* CKB*-canonical model.*
For all formula φ *and for all* CKB*-theory* Γ*,*

$$M_c, \Gamma \models \varphi \textit{ iff } \varphi \in \Gamma.$$

Above, we use Zorn's Lemma to prove:

- $\Box\varphi \notin \Gamma$ implies $\Gamma \not\models \Box\varphi$; and
- $\Diamond\varphi \in \Gamma$ implies $\Gamma \models \Diamond\varphi$.

INTUITIONISTIC EPISTEMIC LOGIC

Artemov and Protopopescu defined a logic IEL such that:

Intuitionistic truth implies intuitionistic knowledge.

IEL consists of

- intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$;
- $T' := K\varphi \to \neg\neg\varphi$;

closed under *modus ponens*.

BHK interpretation

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \to \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- $\neg\varphi$ is an abbreviation for $\varphi \to \bot$.

Artemov and Protopopescu proposed:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

Semantics

An IEL model is a CK-model $M = \langle W, W^{\perp} \preceq, R, V \rangle$ where:

- $W^{\perp} = \emptyset$;
- wRv implies $w \preceq v$;
- $w \preceq v$ implies, for all u, if vRu then wRu;
- for all w there is v such that wRv.

Define:

- $w \models K\varphi$ iff, for all v, wRv implies $v \models \varphi$.

Proposition

If $w \models \varphi$ and $w \preceq v$, then $v \models \varphi$.

As in CK, $w \models \hat{K}\varphi$ holds iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } vRu \text{ and } u \models \varphi.$$

SOME PROPERTIES

- $\mathsf{IEL} \vdash \varphi$ implies $\mathsf{IEL} \vdash K\varphi$;
- $\mathsf{IEL} \vdash K\varphi \to KK\varphi$;
- $\mathsf{IEL} \vdash \neg K\varphi \to K\neg K\varphi$.

Possibility Double Negation

Proposition

For all IEL *model M and world w, if $\hat{K}P$ then $w \models \neg\neg P$.*

Proof.

We have $\neg\neg\varphi$ iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } v \preceq u \text{ and } u \models \varphi.$$

From $R \subseteq \preceq$, we have $\hat{K}P \to \neg\neg P$. $\square$

DOUBLE NEGATION $\rightarrow$ POSSIBILITY

Proposition

For all IEL *model M and world w, if $w \models \neg\neg P$ then $\hat{K}P$.*

Proof.

By contradiction:

- If $\hat{K}P$ fails at w, there is v such that $w \preceq v$ and, for all v', vRv' implies $v' \not\models P$.
- If $\neg\neg P$ holds at w, there is u such that $v \preceq u$ and $u \models P$.
- uR is not empty; fix $u' \in uR$.
- Since $R \subseteq \preceq$, $u' \models P$.
- As $v \preceq u$, $uR \subseteq vR$.
- Therefore $v \preceq u' \not\models P$. $\square$

Possibility — BHK interpretation

Proposition

For all IEL *model* M *and world* w, $\hat{K}P$ *iff* $w \models \neg\neg P$.

Epistemic possibility is impossibility of proof of negation.

Conclusion

Theorem (Das, Marin)

CK *and* IK *do not prove the same* $\Diamond$*-free formulas.*

Theorem (P.)

CKB *and* IKB *prove the same formulas.*

Corollary

CS5 = IS5.

Theorem (P.)

Over IEL, $\Diamond\varphi$ *is equivalent to* $\neg\neg\varphi$.

An Open Problem

Characterize necessary and sufficient conditions for CK-frames to validate the axioms in the modal cube:

- $B_\Box := P \to \Box\Diamond P$, $B_\Diamond := \Diamond\Box P \to P$;
- $4_\Box := \Box\Box P \to \Box P$, $4_\Diamond := \Diamond\Diamond P \to \Diamond P$;
- $5_\Box := \Diamond P \to \Box\Diamond P$, $5_\Diamond := \Diamond\Box P \to \Box P$;
- $T_\Box := \Box P \to P$, $T_\Diamond := P \to \Diamond P$; and
- $D := \Box P \to \Diamond P$.

Characterize necessary and sufficient conditions for CK-frames to validate the axioms:

- $L_{mix} := \Box(\Diamond\neg P \vee P) \to \Box P$;
- $L_\Box := \Box(\Box P \to P) \to \Box P$;
- $L_\Diamond := \Diamond P \to \Diamond(P \wedge \neg\Diamond P)$.

(In general, intuitionistic GL with diamonds is complicated.)

References

- Arisaka, Das, Strassburger, *On Nested Sequents for Constructive Modal Logics*, 2015.
- Artemov, Protopopescu, *Intuitionistic Epistemic Logic*, 2016.
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- de Groot, Shillito, Clouston, *Semantical Analysis of Intuitionistic Modal Logics between CK and IK*, 2024.
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$\sim_c$ IS SYMMETRIC

Suppose $\Gamma \sim_c \Delta$.

- $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Gamma$:
  - Let $\Box\varphi \in \Delta$.
  - Then $\Diamond\Box\varphi \in \Gamma$ as $\Delta \subseteq \Gamma^{\Diamond}$.
  - By $B_{\Diamond}$, $\varphi \in \Gamma$.
- $\Gamma \subseteq \{\varphi \mid \Diamond\varphi \in \Delta\}$.
  - Let $\varphi \in \Gamma$.
  - Then $\Box\Diamond\varphi \in \Gamma$ by $B_{\Diamond}$ and **MP**.
  - Thus $\Diamond\varphi \in \Delta$, as $\Gamma^{\Box} \subseteq \Delta$.

We conclude that $\Delta \sim_c \Gamma$.

$\sim_c$ IS CONFLUENT - I

Suppose $\Gamma \sim_c \Delta \preceq_c \Sigma$. Let Υ be the closure of $\Gamma \cup \{\Diamond\varphi \mid \varphi \in \Sigma\}$ under **MP**. If $\Box\varphi$ is a provable formula in Υ, then $\varphi \in \Sigma$.

- There are formulas $\psi \in \Gamma$ and $\chi_0, \ldots, \chi_n \in \Sigma$ such that

$$\mathsf{CKB} \vdash (\bigwedge_{j<n} \Diamond\chi_j) \wedge \psi \rightarrow \Box\varphi.$$

- By **Nec** and *K*,

$$\mathsf{CKB} \vdash (\bigwedge_{j<n} \Box\Diamond\chi_j) \rightarrow \Box(\psi \rightarrow \Box\varphi)$$

  and so

$$\mathsf{CKB} \vdash (\bigwedge_{j<n} \Box\Diamond\chi_j) \rightarrow (\Diamond\psi \rightarrow \Diamond\Box\varphi).$$

- Since each χ_j is in Σ, so are the $\Box\Diamond\chi_j$, by $B_\Box$.
- Since $\psi \in \Gamma$, $\Diamond\psi \in \Delta$, and thus $\Diamond\psi \in \Sigma$ too.
- By repeated applications of **MP**, we have $\Diamond\Box\varphi \in \Sigma$.
- By $B_\Diamond$, we have $\varphi \in \Sigma$.

$\sim_c$ IS CONFLUENT - II

Suppose $\Gamma \sim_c \Delta \preceq_c \Sigma$. Let Υ be the closure of $\Gamma \cup \{\Diamond\varphi \mid \varphi \in \Sigma\}$ under **MP**.

- $\bot \notin \Upsilon$:
  - Suppose otherwise, then $\Box\bot \in \Upsilon$.
  - So $\bot \in \Sigma$, which is impossible.
- Υ is a set such that: $\Gamma \subseteq \Upsilon$, $\Upsilon^{\Box} \subseteq \Sigma$, $\Sigma \subseteq \Upsilon^{\Diamond}$, and $\bot \notin \Upsilon$.
- Use Zorn's Lemma to extend Υ to a theory Θ with these properties.
- By construction, we have that $\Gamma \preceq_c \Theta \sim_c \Sigma$.