Collapsing Constructive and Intuitionistic Modal Logics

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Introduction

Theorem (Das, Marin)

CK *and* IK *do not prove the same* $\Diamond$*-free formulas:*

- $\mathsf{CK} \nvdash \neg\neg\Box\bot \to \Box\bot$, *and*
- $\mathsf{IK} \vdash \neg\neg\Box\bot \to \Box\bot$

Theorem (P.)

CKB *and* IKB *prove the same formulas.*

The logic CK

CK is the least set of formulas containing:

- intuitionistic tautologies;
- $K_\Box := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_\Diamond := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;

and closed under

$$(\textbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\textbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

The logics CKB, IK, and IKB

Let

- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- $N := \neg\Diamond\bot$;
- $B_\Box := P \to \Box\Diamond P$; and
- $B_\Diamond := \Diamond\Box P \to P$.

Then:

- $\mathsf{CKB} := \mathsf{CK} + \{B_\Box, B_\Diamond\}$;
- $\mathsf{IK} := \mathsf{CK} + \{FS, DP, N\}$; and
- $\mathsf{IKB} := \mathsf{IK} + \{B_\Box, B_\Diamond\} = \mathsf{CKB} + \{FS, DP, N\}$.

CK-MODELS

A CK-model is a tuple $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $W^{\perp} \subseteq W$ is the set of *fallible worlds*;
- the *intuitionistic relation* $\preceq$ is a reflexive and transitive relation over W;
- the *modal relation* R is a relation over W;
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a *valuation function*.

We require:

- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- for all $P \in \mathrm{Prop}$, $W^{\perp} \subseteq V(P)$;
- if $w \in W^{\perp}$ and either $w \preceq v$ or wRv, then $v \in W^{\perp}$.

IK-models

An IK-model is a CK-model where:

- $W^{\perp} = \emptyset$;
- R is forward and backward confluent:

$$\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[d, "R"'] & w' \arrow[d, dashed, "R"] \\
v \arrow[r, dashed, "\preceq"'] & v'
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, dashed, "\preceq"] \arrow[d, "R"'] & w' \arrow[d, dashed, "R"] \\
v \arrow[r, "\preceq"'] & v'
\end{tikzcd}$$

An IKB-model is an IK-model where R is symmetric.

Valuation

- $M, w \models P$ iff $w \in V(P)$;
- $M, w \models \bot$ iff $w \in W^{\bot}$;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \to \psi$ iff, for all $v \in W$, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$; and
- $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ then, there is u such that vRu and $M, u \models \varphi$.

Main Theorem

Theorem

For all modal formula φ, the following are equivalent:

1. $\mathsf{CKB} \vdash \varphi$;
2. $\mathsf{IKB} \vdash \varphi$; *and*
3. $\mathsf{IKB} \models \varphi$.

$$\begin{tikzcd}
\mathsf{CKB} \vdash \varphi \arrow[rr, "\text{By definition.}"] & & \mathsf{IKB} \vdash \varphi \arrow[dl, "\text{Soundness}"] \\
& \mathsf{IKB} \models \varphi \arrow[ul, "\text{Canonical Model}"] &
\end{tikzcd}$$

Symmetry Implies Confluences Coincide

Lemma

Let M be a CK*-model where the modal relation $\sim$ is symmetric. Then $\sim$ is forward confluent iff $\sim$ is backward confluent.*

$$
\begin{tikzcd}
w \arrow[r, "\preceq"] \arrow[d, leftrightarrow, "\sim"'] & w' \arrow[d, leftrightarrow, dashed, "\sim"] \\
v \arrow[r, dashed, "\preceq"'] & v'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
w \arrow[r, dashed, "\preceq"] \arrow[d, leftrightarrow, "\sim"'] & w' \arrow[d, leftrightarrow, dashed, "\sim"] \\
v \arrow[r, "\preceq"'] & v'
\end{tikzcd}
$$

Symmetry Implies Confluence is Necessary

Lemma

There is a CK*-model* $M = \langle W, W^{\perp}, \preceq, \sim, V \rangle$ *and* $w \in W$ *such that:*

- $\sim$ *is a symmetric relation;*
- $B_{\Box} := P \to \Box\Diamond P$ *does not hold at* w.

$$\begin{tikzcd}
w \models P \arrow[d, leftrightarrow, "\sim"'] & \\
v \arrow[r, "\preceq"] & v'
\end{tikzcd}$$

Existing Results

Theorem (Arisaka, Das, Straßburger)

$\mathsf{CKB} \vdash DP$ *and* $\mathsf{CKB} \vdash N$.

Theorem (De Groot, Shillito, Clouston)

Let $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ *be a* CK*-model. Then:*

- *Suppose that, for all* $w, v \in W$*,* wRv*, and* $v \in W^{\perp}$ *implies* $w \in W^{\perp}$*. Then* $M \models N$*.*
- *Suppose that* R *is forward and backward confluent. Then* $M \models DP$ *and* $M \models FS$*.*

A Canonical Model for CKB

A (consistent) CKB-theory Γ is a set of formulas such that:

- Γ contains all the axioms of CKB and is closed under **MP**;
- if $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$;
- $\bot \notin \Gamma$.

Definition

The CKB-canonical model is $M_c := \langle W_c, W_c^{\bot}, \preceq_c, \sim_c, V_c \rangle$ where:

- $W_c := \{\Gamma \mid \Gamma \text{ is a CKB-theory}\}$;
- $W_c^{\bot} = \emptyset$;
- $\Gamma \preceq_c \Delta$ iff $\Gamma \subseteq \Delta$;
- $\Gamma \sim_c \Delta$ iff $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$ and $\Delta \subseteq \{\varphi \mid \Diamond\varphi \in \Gamma\}$;
- $\Gamma \in V_c(P)$ iff $P \in \Gamma$.

Truth Lemma

Lemma

The CKB*-canonical model* M_c *is an* IKB*-model.*

The following lemma uses standard techniques:

Lemma

Let M_c *be the* CKB*-canonical model.*
For all formula φ *and for all* CKB*-theory* Γ*,*

$$M_c, \Gamma \models \varphi \textit{ iff } \varphi \in \Gamma.$$

Above, we use Zorn's Lemma to prove:

- $\Box\varphi \notin \Gamma$ implies $\Gamma \not\models \Box\varphi$; and
- $\Diamond\varphi \in \Gamma$ implies $\Gamma \models \Diamond\varphi$.

Conclusion

While, in general, constructive and intuitionistic versions of the same logic do not coincide, we have:

Theorem (P.)

CKB *and* IKB *prove the same formulas.*

This result extends to logics proving the axiom B. For example:

Corollary

$\mathsf{CS5} = \mathsf{IS5}$.

An Open Problem

Characterize necessary and sufficient conditions for CK-frames to validate the axioms in the modal cube:

- $B_\Box := P \to \Box\Diamond P$, $B_\Diamond := \Diamond\Box P \to P$;
- $4_\Box := \Box P \to \Box\Box P$, $4_\Diamond := \Diamond\Diamond P \to \Diamond P$;
- $5_\Box := \Diamond P \to \Box\Diamond P$, $5_\Diamond := \Diamond\Box P \to \Box P$;
- $T_\Box := \Box P \to P$, $T_\Diamond := P \to \Diamond P$; and
- $D := \Box P \to \Diamond P$.

Characterize necessary and sufficient conditions for CK-frames to validate the axioms:

- $L_{mix} := \Box(\Diamond\neg P \vee P) \to \Box P$;
- $L_\Box := \Box(\Box P \to P) \to \Box P$;
- $L_\Diamond := \Diamond P \to \Diamond(P \wedge \neg\Diamond P)$.

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