Higher-order feedback computation

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18 March 2024

Available at: leonardopacheco.xyz/slides/suugakukai2024-hofb.pdf

Feedback Turing Machines

Feedback machines have access to information on convergence/divergence of feedback machines.

Feedback Oracle

Feedback Turing machines have access to a halting oracle:

$$h(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^h(n) \text{ converges;} \\ \uparrow, & \text{if } \{e\}^h(n) \text{ diverges;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

When $h(e,n)$ is undefined, the computation $\{e\}^h(n)$ *freezes*.

FREEZING

Let e be such that

$$e(n) := \begin{cases} \text{diverges}, & \text{if } \{n\}^h(n) \text{ converges;} \\ 0, & \text{if } \{n\}^h(n) \text{ diverges.} \end{cases}$$

Then

$$e(e) \text{ converges} \iff e(e) \text{ diverges.}$$

Therefore $e(e)$ freezes.

CHARACTERIZATION

Theorem (Ackerman, Freer, Lubarsky)

The following classes coincide:

1. *the feedback semi-computable sets;*
2. *the Π^1_1 sets;*
3. *the sets definable by arithmetic inductive operators; and*
4. *the sets of winning positions of Gale–Stewart games whose payoffs are Σ^0_1.*

SECOND-ORDER FEEDBACK

2-feedback Turing machines have access to 2 freezing oracles:

$$f_0(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{f_0,f_1}(n) \text{ converges;} \\ \uparrow_0, & \text{if } \{e\}^{f_0,f_1}(n) \text{ diverges;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

$$f_1(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{f_0,f_1}(n) \text{ converges;} \\ \uparrow_0, & \text{if } \{e\}^{f_0,f_1}(n) \text{ diverges;} \\ \uparrow_1, & \text{if } \{e\}^{f_0,f_1}(n) \text{ freezes;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Higher-Order Feedback

Fix $\alpha < \omega_1^{\text{ck}}$.
For $\beta < \alpha$, let

$$f_\beta(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{\{f_\gamma\}_{\gamma<\alpha}}(n) \text{ converges;} \\ \uparrow_{\beta'}, & \text{if } \{e\}^{\{f_\gamma\}_{\gamma<\alpha}}(n)\ \beta'\text{-freezes } (\beta' < \beta); \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Characterization

Theorem (Aguilera, Lubarsky, P.)

For all $\alpha < \omega_1^{\mathrm{ck}}$, the following classes coincide:

1. *the $(\alpha + 1)$-feedback semi-computable sets;*
2. *the sets definable by $\alpha + 1$ simultaneous arithmetical inductive operators; and*
3. *the sets of winning positions of Gale–Stewart games whose payoffs are differences of α many Σ^0_2 sets.*

Future work

Connection between feedback and reflecting ordinals:

Almost a Theorem

For all $\alpha < \omega_1^{ck}$, the following classes coincide:

1. *the $(\alpha + 1)$-feedback semi-computable sets, and*
2. *the Σ_1-definable sets in $L_{\beta_{\alpha+1}}$, where $\beta_{\alpha+1}$ is the least $\alpha + 1$-reflecting ordinal.*

There are strict and loose notions of feedback hyperjump.
The following follows from work of Aguilera and Lubarsky:

Theorem

A set of natural numbers is 2-feedback semi-computable iff it is reducible to the loose feedback hyperjump $\mathcal{LO}$.

The relation between higher-order feedback and strict feedback hyperjump is unclear.

References

[1] Ackerman, Freer, Lubarsky, "An Introduction to Feedback Turing Computability", 2020.

[2] Aguilera, Lubarsky, Pacheco, "Higher-order feedback computability", submitted.

[3] Rogers Jr., "Theory of Recursive Functions and Effective Computability", 1967.