Non-classical variations of GL

Leonardo Pacheco
TU Wien

(ongoing work with Grigorii Stepanov)

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Introduction

modal logic = propositional logic + $\Box$ + $\Diamond$.

- Two main non-classical varieties:
  - constructive modal logic, and
  - intuitionistic modal logic.
- I want to axiomatize constructive and intuitionistic versions of the logic GL.
- A proof theoretic characterization of IGL was given by Das, van der Giessen and Marin.

An axiomatization for constructive GL?

Axioms

- all intuitionistic tautologies;
- $K_{\Box} := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_{\Diamond} := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;
- $4_{\Box} := \Box\varphi \to \Box\Box\varphi$;
- $4_{\Diamond} := \Diamond\Diamond\varphi \to \Diamond\varphi$; and
- $L_{\Box} := \Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Rules:

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

SEMANTICS FOR CONSTRUCTIVE GL?

Bi-relational Kripke models $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $W^{\perp} \subseteq W$ is the set of *fallible worlds*;
- $\preceq$ is a reflexive and transitive relation over W;
- $\sqsubset$ is a transitive relation over W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a valuation function.

We require that:

- $W^{\perp} \subseteq V(P)$;
- $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- there is no infinite sequence $w_0 \preceq ; \sqsubset w_1 \preceq ; \sqsubset w_2 \preceq ; \sqsubset \cdots$;
- for all $w \sqsubset v \preceq v'$, there is w' such that $w \preceq w' \sqsubset v'$.

VALUATIONS

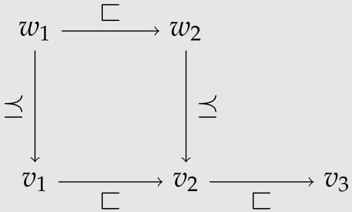
Given $M = \langle W, W^{\perp}, \preceq, R, V\rangle$, define

$$\|\Box\varphi\|^{M} := \{w \mid \text{for all } v \text{ and } w\text{, if } w \preceq v \text{ and } v \sqsubset u\text{, then } u \in \|\varphi\|^{M}\}$$

$$\|\Diamond\varphi\|^{M} := \{w \mid \text{for all } v \text{ if } w \preceq v\text{, then there is } v \sqsubset u \text{ such that } u \in \|\varphi\|^{M}\}$$

THE DUAL OF LÖB'S THEOREM

$L_\Diamond := \Diamond P \to \Diamond(P \wedge \Box\neg P)$ of fails at w_1:



(P holds everywhere.)

Failure to prove the completeness

Proofs of completeness using finitary canonical models seem to need some diamond version of

$$L_\Box := \Box(\Box\varphi \to \varphi) \to \Box\varphi.$$

Two questions:

- Is the axiomatization described above complete?
- If not, what extra axiom is needed?

References

[1] Das, van der Giessen, Marin, “Intuitionistic Gödel–Löb Logic, à La Simpson: Labelled Systems and Birelational Semantics”, 2024.

[2] Balbiani, Dieguez, Fernández-Duque, “Some Constructive Variants of S4 with the Finite Model Property”, 2021.