A constructive variation of GL

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We describe a constructive variation of the provability logic GL. The logic CGL is based on Mendler and de Paiva's constructive modal logic CK [4]. The system CGL is obtained by adding Löb's axiom $\Box(\Box P \to P) \to \Box P$ to CK; the CGL models are obtained by adding a reverse well-foundedness condition to the CK models. We show:

Theorem. The constructive modal logic CGL is complete over CGL models.

We also describe ongoing work on non-classical varieties of GL based on the intuitionistic modal logic IK [5] and the Gödel modal logic GK [3].

The axioms of CGL are: all intuitionistic tautologies; $K_\Box := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$; $K_\Diamond := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$; $4_\Box := \Box\varphi \to \Box\Box\varphi$; $4_\Diamond := \Diamond\Diamond\varphi \to \Diamond\varphi$; and $L := \Box(\Box\varphi \to \varphi) \to \Box\varphi$. CGL is closed under necessitation and modus ponens:

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \qquad \text{and} \qquad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

A CGL model is a bi-relational Kripke model $F = \langle W, W^\perp, \preceq, \sqsubset, V\rangle$ where: W is the set of possible worlds; $W^\perp$ is the set of fallible worlds, where the proposition `false` is true; $\preceq$ is a partial order, representing the increase of information in the model; $\sqsubset$ a transitive relation; and V is a valuation. We add the following requirements:

- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- if $w \sqsubset v \preceq v'$, then there is w' such that $w \preceq w' \sqsubset v'$; and
- the relation $\preceq;\sqsubset$ restricted to $W \setminus W^\perp$ is reverse well-founded.

Here, $\preceq;\sqsubset$ is the composition of the relations $\preceq$ and $\sqsubset$.

Given a formula φ, we define a finite canonical CGL model M_φ and show a Truth Lemma for M_φ: for all world Γ of M_φ and subformula ψ of φ,

$$\Gamma \models \psi \text{ iff } \psi \in \Gamma.$$

These canonical models allow us to prove the completeness of CGL over CGL models. Our construction is based on the constructions for GL by Boolos [2] and for $\mathsf{CS4}$ by Balbiani, Dieguez, and Fernández-Duque [1].

References

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