A Short Overview of Non-classical Modal Logics

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A white lie

Modal logic =

- propositional logic
- + $\Box$: necessity
- + $\Diamond$: possibility

SOME READINGS FOR $\Box$ AND $\Diamond$

Necessity:

- φ is known.
- φ is provable.
- φ always holds.
- The interior of the set defined by φ.
- φ is true in all forcing extensions.

Possibility:

- φ is epistemically possible.
- φ is consistent.
- φ holds sometime in the future.
- The closure of the set defined by φ.
- φ is true in some forcing extension.

And many more.

Diamonds are not real

Over the modal logic K,

$$\Diamond\varphi \equiv \neg\Box\neg\varphi.$$

Our setting

Non-classical modal logic =

- *intuitionistic* propositional logic
- + $\Box$: necessity
- + $\Diamond$: possibility

Why?

- Finer distinctions than classical setting.
- Some new possibilities appear.
- Many mathematical challenges still available.

Outline

- CK versus IK
- iK4 versus IK4
- coreflection and IEL
- fully labeled proof systems

CK — CONSTRUCTIVE K

- First defined by Wijesekera[1].
- Axioms:
  - $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$
- Rules:
  - $\varphi \to \psi, \varphi \vdash \psi$
  - $\varphi \vdash \Box\varphi$

[1] "Constructive modal logics I", 1990.

IK — intuitionistic K

- First defined by Fischer-Servi[2].
- Axioms:
  - $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$
  - $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$
  - $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$
  - $\neg\Diamond\bot$
- Same rules as CK

[2] "On modal logics with an intuitionistic base", 1977.

CK VERSUS IK

While the extra axioms of IK all involve $\Diamond$s, the $\Diamond$-free fragments of CK and IK do not coincide:

Theorem (Das, Marin)

CK *and* IK *do not prove the same* $\Diamond$*-free formulas:*

- CK $\nvdash \neg\neg\Box\bot \to \Box\bot$, *and*
- IK $\vdash \neg\neg\Box\bot \to \Box\bot$

On the other hand, with the axioms $\Diamond\varphi \to \Box\Diamond\varphi$ and $\Diamond\Box\varphi \to \Box\varphi$, the extensions of CK and IK coincide:

Theorem (P.)

CKB *and* IKB *prove the same formulas.*

I'll sketch the first result today.

CK-MODELS

A CK-model is a tuple $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ where:

- W is the set of *possible worlds*;
- $W^{\perp} \subseteq W$ is the set of *fallible worlds*;
- the *intuitionistic relation* $\preceq$ is a reflexive and transitive relation over W;
- the *modal relation* R is a relation over W;
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a *valuation function*.

We require:

- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- for all $P \in \mathrm{Prop}$, $W^{\perp} \subseteq V(P)$;
- if $w \in W^{\perp}$ and either $w \preceq v$ or wRv, then $v \in W^{\perp}$.

VALUATION

- $M, w \models P$ iff $w \in V(P)$;
- $M, w \models \bot$ iff $w \in W^{\bot}$;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \rightarrow \psi$ iff, for all $v \in W$, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$; and
- $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ then, there is u such that vRu and $M, u \models \varphi$.

IK-MODELS

An IK-model is a CK-model where:

- $W^{\perp} = \emptyset$;
- R is forward and backward confluent:

$$\begin{tikzcd} w' \arrow[r, dashed, "R"] & v' \\ w \arrow[u, "\preceq"] \arrow[r, "R"'] & v \arrow[u, dashed, "\preceq"'] \end{tikzcd} \qquad \begin{tikzcd} w' \arrow[r, dashed, "R"] & v' \\ w \arrow[u, dashed, "\preceq"] \arrow[r, "R"'] & v \arrow[u, "\preceq"'] \end{tikzcd}$$

CK $\not\models \neg\neg\Box\bot \to \Box\bot$

Consider the model below:

$$\begin{tikzcd} w_2 & \\ w_0 \arrow[u, "\preceq"] \arrow[r, "R"'] & w_1 \end{tikzcd}$$

We have that $w_0 \models \neg\neg\Box\bot$ but $w_0 \not\models \Box\bot$.

$$(w \models \neg\neg\Box\bot \text{ iff } \forall v \succeq w \exists u \succeq v . u \models \Box\bot)$$

IK $\models \neg\neg\Box\bot \to \Box\bot$

Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.

$$\begin{tikzcd} v_1 \arrow[r, "R"] & v_2 \\ w \arrow[u, "\preceq"] & \end{tikzcd}$$

$$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v. u \not\models \Box\bot)$$

IK $\models \neg\neg\Box\bot \to \Box\bot$

Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.

$$
\begin{tikzcd}
u_1 & \\
v_1 \arrow[u, "\preceq"] \arrow[r, "R"] & v_2 \\
w \arrow[u, "\preceq"] &
\end{tikzcd}
$$

$$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v . u \not\models \Box\bot)$$

IK $\models \neg\neg\Box\bot \rightarrow \Box\bot$

Suppose $w \not\models \Box\bot$, then $w \not\models \neg\neg\Box\bot$.

$$\begin{tikzcd} u_1 \arrow[r, dashed, "R"] & u_2 \\ v_1 \arrow[u, "\preceq"] \arrow[r, "R"] & v_2 \arrow[u, dashed, "\preceq"'] \\ w \arrow[u, "\preceq"] \end{tikzcd}$$

$(w \not\models \neg\neg\Box\bot \text{ iff } \exists v \succeq w \forall u \succeq v.u \not\models \Box\bot)$

Topological Semantics

The following theorems illustrate how having both $\Box$s and $\Diamond$s might increase the complexity of the logic.

Theorem (Montacute, P.)

iK4 *is complete with respect to uni-topological* iK4*-models.*

Theorem (Aguilera, Fernández-Duque, P.[3])

IK4 *is complete with respect to tri-topological* iK4*-models.*

These generalize classical theorems by McKinsey–Tarski and Esakia. The first extends work by de Groot and Shillito[4].

[3] "Polytopological Semantics for Intuitionistic Modal Logics ", arXiv:2604.23234.

[4] "Intuitionistic S4 as a logic of topological spaces", 2025.

TOPOLOGICAL MODELS FOR iK4

An topological iK4-model is a tuple $M = \langle W, \preceq, \tau, V \rangle$ where:

- W is a non-empty set;
- the $\preceq$ is the specialization order of τ;
- τ is a topology on W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a *valuation function*.

We further require that,

- for every $X \subseteq W$, $e_\tau(X) \in \tau$ is upwards closed under $\preceq$.
- τ is T_d: for every w, there is U, V such that $\{w\} = U \cap V^c$.

Define the valuation as follows:

- $\|\Box\varphi\| = \bar{d}_\tau(\|\varphi\|)$, where $\bar{d}_\tau$ is the coderivative of τ.

Note: we can omit $\preceq$!

TRITOPOLOGICAL MODELS FOR CK4

A tritopological CK4-model is a tuple $\langle W, \tau_{\rightarrow}, \tau_{\Diamond}, \tau_{\Box}, V\rangle$, where:

- W is a non-empty set;
- $\tau_{\rightarrow}$ is a topology over W;
- $\tau_{\Diamond}, \tau_{\Box}$ are T_d-topologies over W;
- $\bar{d}_{\Box} i_{\rightarrow} = i_{\rightarrow} \bar{d}_{\Box} i \rightarrow$ and $e_{\Box} i_{\rightarrow} = i_{\Diamond} \bar{d}_{\Box} i_{\rightarrow}$.

Define the valuation as follows:

- $\|\varphi \rightarrow \psi\| = i_{\rightarrow}(W \setminus \|\varphi\| \cup \|\psi\|)$;
- $\|\Box\varphi\| = \bar{d}_{\Box}(\|\varphi\|)$;
- $\|\Diamond\varphi\| = i_{\rightarrow}(d_{\Diamond}\|\varphi\|)$.

TWO IS NOT ENOUGH

Theorem

Let $M = \langle W, \tau_{\rightarrow}, \tau_{\Diamond}, \tau_{\Box}, V\rangle$ be a CK4*-model where $\tau_{\Box}$ is the meet of $\tau_{\rightarrow}$ and $\tau_{\Diamond}$, that is, the greatest topology contained in the intersection $\tau_{\rightarrow} \cap \tau_{\Diamond}$. Then*

$$M, w \models \Box Q \rightarrow P \vee (P \rightarrow Q).$$

This is not a theorem of CK4*.*

$\Box Q \to P \vee (P \to Q)$ IS VALID IF $\tau_\Box \subseteq \tau_\to$

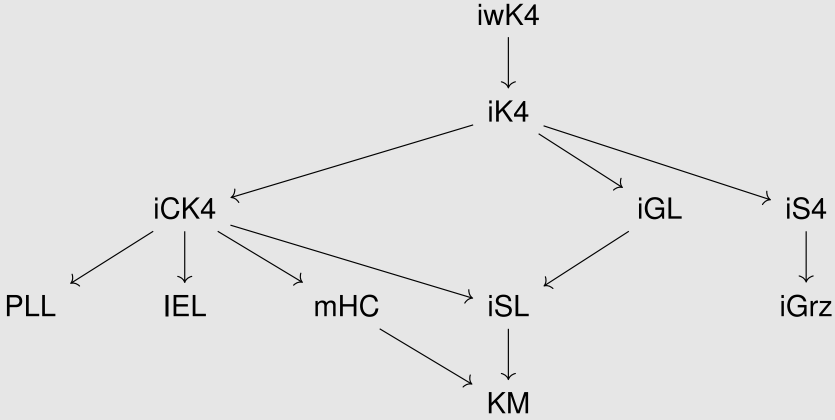
- Suppose that $x \in \|\Box Q\| \setminus \|P\|$.
- x has a $\tau_\Box$-neighborhood U such that $U \setminus \{x\} \subseteq \|Q\|$.
- By assumption, U is also a $\tau_\to$-neighborhood of x.
- Since $x \notin \|P\|$, $U \cap \|P\| \subseteq U \setminus \{x\} \subseteq \|Q\|$, hence U witnesses that $x \in \|P \to Q\|$.
- Since x was arbitrary, $\|\Box Q \to P \vee (P \to Q)\| = W$.

$\Box Q \to P \vee (P \to Q)$ IS NOT VALID OVER BIRELATIONAL MODELS

$$
\begin{tikzcd}
v \arrow[dr, "\sqsubset"] & \\
& u \\
w \arrow[uu, "\preceq"] \arrow[ur, "\sqsubset"'] &
\end{tikzcd}
$$

P holds at v, Q holds at u.

Unimodal Logics — $\Box$ only



jww Yoàv Montacute
preprint out soon

Bimodal Logics — $\Box$ and $\Diamond$

$$
\begin{tikzcd}
\mathsf{CK4} \arrow[r] \arrow[d] & \mathsf{CS4} \arrow[d] \\
\mathsf{IK4} \arrow[r] \arrow[d] & \mathsf{IS4} \arrow[d] \\
\mathsf{GK4} \arrow[r] \arrow[d] & \mathsf{GS4} \arrow[d] \\
\mathsf{GK4^c} \arrow[r] & \mathsf{GS4^c}
\end{tikzcd}
$$

jww Juan Aguilera and David Fernández-Duque
arXiv:2604.23234

INTUITIONISTIC EPISTEMIC LOGIC

Artemov and Protopopescu defined a logic IEL such that:

Intuitionistic truth implies intuitionistic knowledge.

IEL consists of

- intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$;
- $T' := K\varphi \to \neg\neg\varphi$;

closed under *modus ponens*.

BHK INTERPRETATION

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \to \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- $\neg\varphi$ is an abbreviation for $\varphi \to \bot$.

Artemov and Protopopescu proposed:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

KNOWLEDGE AS NECESSITY

The standard semantics for IEL interprets K as a $\Box$:

$$M, w \models K\varphi \text{ iff, for all } v \in w, \text{ if } wRv, \text{ then } M, v \models \varphi.$$

We can show

Proposition

Over IEL*, if K is interpret as a $\Box$, then*

$$\Diamond\varphi \equiv \neg\neg\varphi.$$

So epistemic possibility is classical truth.

Knowledge as Possibility

We can also work with a semantics where K as a $\Diamond$:

$M, w \models K\varphi$ iff, for all $v \in W$,
if $w \preceq v$, then there is u such that vRu and $M, u \models \varphi$.

Then:

Proposition

Over IEL*, if K is interpret as a $\Diamond$, then*

$$\Box\varphi \equiv \varphi.$$

So epistemic necessity is intuitionistic truth.

This is jww with Igor Sedlár.

Other coreflection logics

- iCK4 $:=$ iK $+ \varphi \to \Box\varphi$;
- IEL $:=$ iCK4 $+ \Box\varphi \to \neg\neg\varphi$;
- PLL $:=$ iCK4 $+ \Box\Box\varphi \to \Box\varphi$;
- mHC $:=$ iCK4 $+ \Box\varphi \to ((\psi \to \varphi) \vee \psi)$;
- iSL $:=$ iCK4 $+ \Box(\Box\varphi \to \varphi) \to \Box\varphi$;
- KM $:=$ mHC $+ \Box(\Box\varphi \to \varphi) \to \Box\varphi$;

Canonical models in classical modal logic

- Compactness fails for both GL and μ-calculus.
- Therefore the truth lemma fails for them.
- We can overcome this:
  - finite canonical models for GL,
  - final canonical models for μ-calculus (for $\mathsf{wK4}$ models).

CANONICAL MODELS IN INTUITIONISTIC MODAL LOGIC

- Use theories Γ:
  - $\Gamma \vdash_L \varphi$ implies $\varphi \in \Gamma$,
  - $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.
- Works for *completeness*.
- Usually breaks down for decidability:
  - CS4 needs a complicated filtration argument,[5]
  - IS4 needs other methods.[6]

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[5] Balbiani, Diéguez, Fernández-Duque, "Some constructive variants of S4 with the finite model property", 2021.

[6] Girlando, Kuznets, Marin, Morales, Straßburger, "Intuitionistic S4 is decidable", 2023. Note: the proof in this paper is wrong.

LABELED SEQUENTS

~~We now shift our focus from semantics to proof theory.~~

- Labeled formulas:

$$x : \varphi$$

- Labeled sequents:

$$\mathbf{R}, \Gamma \vdash \Delta,$$

  where **R** is a collection of statements of the forms
  - $x \preceq y$,
  - $x' R y'$.

In this talk, I'll focus on proof systems for IGL and the μ-calculus.

Every rule is invertible

$$\to\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \to \psi}\ (y \text{ fresh})$$

$$\Box\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, yRz, \Gamma \vdash \Delta, z : \varphi}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box\varphi}\ (y, z \text{ fresh})$$

IGL proves $\Box(\Box P \to P) \to \Box P$

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P$}
\AxiomC{$\vdots$}
\UnaryInfC{$xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$}
\LeftLabel{tr}
\UnaryInfC{$xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$}
\LeftLabel{$\Box$l}
\UnaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P$}
\LeftLabel{$\to$l}
\BinaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P$}
\LeftLabel{$\Box$l}
\UnaryInfC{$xRy, x : \Box(\Box P \to P) \vdash y : P$}
\LeftLabel{$\Box$r}
\UnaryInfC{$x : \Box(\Box P \to P) \vdash x : \Box P$}
\LeftLabel{$\to$r}
\UnaryInfC{$\vdash x : \Box(\Box P \to P) \to \Box P$}
\end{prooftree}
$$

Non-wellfounded proofs

- Allow (good) infinite paths in proofs:
  - for GL, require $\{x_i\}_{i \in \omega}$ with $x_i R x_{i+1}$.
  - for μ-calculus, require $\{x_i : \mu^{\alpha_i} X.\varphi\}$ with $\alpha_i > \alpha_{i+1}$.
- Good for completeness.
- Not good for decidability.

IGL proves $\Box(\Box P \to P) \to \Box P$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{id}
\UnaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P$}
\AxiomC{$xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$}
\LeftLabel{tr}
\UnaryInfC{$xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P$}
\LeftLabel{$\Box$l}
\UnaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P$}
\LeftLabel{$\to$l}
\BinaryInfC{$xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P$}
\LeftLabel{$\Box$l}
\UnaryInfC{$xRy, x : \Box(\Box P \to P) \vdash y : P$}
\LeftLabel{$\Box$r}
\UnaryInfC{$x : \Box(\Box P \to P) \vdash x : \Box P$}
\LeftLabel{$\to$r}
\UnaryInfC{$\vdash x : \Box(\Box P \to P) \to \Box P$}
\end{prooftree}
$$

$$x \mapsto = x \text{ and } y \mapsto z.$$

SOUNDNESS

- $\vdash_L \varphi$ and $\not\models_L \varphi$.
- Find a path on the proof of φ which is (stepwise) refuted.
- The path condition will imply a contradiction.

COMPLETENESS — NON-WELLFOUNDED SYSTEMS

- By a proof(model?)-search argument.
- Keep "saturating" the leaves of the partial proof tree.
- For example,

$$\to\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, x : \varphi \to \psi, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \to \psi}\ (y \text{ fresh})$$

  where there is no z such that $x \preceq z$ is not in the lower sequent and $z : \varphi$ is not on the right of the lower sequent.
- If we ensure every such non-saturated formula is saturated, we get either a proof or a counterexample.

COMPLETENESS — NON-WELLFOUNDED SYSTEMS

- By a proof-search argument.
- Keep “saturating” the leaves of the partial proof tree, in two alternating phases:
  - only rules which do not introduce statements $x \preceq y$,
  - only rules which introduce statements $x \preceq y$.
- Do some loop-checking to guarantee this search stops:
  - either a good loop to a lower sequent exists,
  - or we can simulate a layer in a lower one.

SOME RESULTS

Theorem

- *(Das, van der Giessen, Marin[7]) ℓIGL is complete w.r.t. IGL-models.*
- *(Aguilera, P.[8]) cmℓIGL is complete w.r.t. IGL-models.*
- *(Lyon, P.) IGL is decidable.*

Theorem (P.)

- *labμCK$_\omega$ is sound and complete w.r.t. CK-models.*
- *labμIK$_\omega$ is sound and complete w.r.t. IK-models.*

Furthermore, the three logics above are decidable.

Open: Hilbert-style proof systems.

[7] "Intuitionistic Gödel-Löb logic, à la Simpson", 2023.
[8] "Intuitionistic Gödel-Löb without Sharps", 2025.

Some things I didn't talk about

- Intuitionistic provability logics (e.g., the provability logic of HA).
- Your favorite version of interpolation.
- Computational complexity.
- Non-labeled proof theory.
- Relations to type theory.
- Subintuitionistic and intermediate modal logics.

Thank you!