Epistemic possiblity in intuitionistic epistemic logic

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Modal Logic

Modal logic is

$$\text{propositional logic} + \Box + \Diamond$$

In epistemic logic:

- $K\varphi$:= it is known that φ;
- $\hat{K}\varphi$:= it is epistemically possible(?) that φ.

CLASSICAL EPISTEMIC LOGIC

Axioms:

- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $T := K\varphi \to \varphi$;
- $4 := K\varphi \to KK\varphi$;
- $5 := \neg K\varphi \to K\neg K\varphi$.

Rules:

- $MP := \varphi \to \psi, \varphi \vdash \psi$;
- $Nec := \varphi \vdash K\varphi$.

Models:

- Kripke models $\langle W, R, V \rangle$ where R is an equivalence relation.

INTUITIONISTIC EPISTEMIC LOGIC

Artemov and Protopopescu defined a logic IEL such that:

Intuitionistic truth implies intuitionistic knowledge.

IEL consists of

- intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$;
- $T' := K\varphi \to \neg\neg\varphi$;

closed under *modus ponens*.

BHK INTERPRETATION

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \to \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- $\neg\varphi$ is an abbreviation for $\varphi \to \bot$.

Artemov and Protopopescu proposed:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

SEMANTICS

An IEL model is a tuple $M = \langle W, \preceq, R, V \rangle$ where:

- $\preceq$ is a preorder on W;
- V is monotone w.r.t. $\preceq$;
- wRv implies $w \preceq v$;
- $w \preceq v$ implies, for all u, if vRu then wRu;
- for all w there is v such that wRv.

Define:

- $w \models K\varphi$ iff, for all v, wRv implies $v \models \varphi$.

Proposition

If $w \models \varphi$ and $w \preceq v$, then $v \models \varphi$.

Properties — Necessitation

Proposition

IEL *is closed under necessitation:*

$$\mathsf{IEL} \vdash \varphi \textit{ implies } \mathsf{IEL} \vdash K\varphi.$$

Properties — Introspection

Proposition

IEL *proves positive introspection:*

$$K\varphi \to KK\varphi.$$

Proposition

IEL *proves negative introspection:*

$$\neg K\varphi \to K\neg K\varphi.$$

POSSIBILITY — SEMANTICS

$w \models \hat{K}\varphi$ holds iff

for all $v \succeq w$, there is u such that vRu and $u \models \varphi$.

Proposition

If $w \models \hat{K}\varphi$ and $w \preceq v$, then $v \models \hat{K}\varphi$.

Possibility — Double Negation

Proposition

For all IEL *model M and world w, if $\hat{K}P$ then $w \models \neg\neg P$.*

Proof.

We have $\neg\neg\varphi$ iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } v \preceq u \text{ and } u \models \varphi.$$

From $R \subseteq \preceq$, we have $\hat{K}P \rightarrow \neg\neg P$. □

Possibility — Double Negation (Reversal)

Proposition

For all IEL *model* M *and world* w, *if* $w \models \neg\neg P$ *then* $\hat{K}P$.

Proof.

By contradiction:

- If $\hat{K}P$ fails at w, there is v such that $w \preceq v$ and, for all v', vRv' implies $v' \not\models P$.
- If $\neg\neg P$ holds at w, there is u such that $v \preceq u$ and $u \models P$.
- uR is not empty; fix $u' \in uR$.
- Since $R \subseteq \preceq$, $u' \models P$.
- As $v \preceq u$, $uR \subseteq vR$.
- Therefore $v \preceq u' \not\models P$. $\square$

Possibility — BHK interpretation

Proposition

For all IEL *model M and world w, $\hat{K}P$ iff $w \models \neg\neg P$.*

Epistemic possibility is impossibility of proof of negation.

Thank you!

For more details on intuitionistic epistemic logic, see

- Artemov, Protopopescu, "Intuitionistic Epistemic Logic".

For more details on intuitionistic modal logics, see

- van der Giessen, "Uniform Interpolation and Admissible Rules: Proof-theoretic investigations into (intuitionistic) modal logics ".