The μ-calculus' Alternation Hierarchy is Strict over Non-Trivial Fusion Logics

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THE μ-CALCULUS

modal μ-calculus = modal logic + fixed-point operators

- μ: least fixed-point operator
- ν: greatest fixed-point operator

ALTERNATION DEPTH

The valuation of νX and μY depend on each other:

$$\nu X.\, \underbrace{\mu Y.\, \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

Some results on the unimodal μ-calculus

Theorem (Bradfield [2])

The μ-calculus alternation hierarchy is strict over all frames.

Theorem (Alberucci–Facchini [1])

The μ-calculus alternation hierarchy collapses to the alternation-free fragment over transitive frames.

Theorem (Alberucci–Facchini [1])

The μ-calculus alternation hierarchy collapses to modal logic over equivalence relations.

For a survey, see [4].

OUR RESULT — SIMPLIFIED

The fusion $\mathsf{S5} \otimes \mathsf{S5}$ has two independent pairs of modalities $\Box_0/\Diamond_0$ and $\Box_1/\Diamond_1$, each satisfying $\mathsf{S5}$.

Theorem

The μ-calculus' alternation hierarchy is strict over $\mathsf{S5} \otimes \mathsf{S5}$.

The same result holds for the fusion of any two *non-trivial* logics.

DEFINITIONS

The μ-formulas are defined by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi_i \mid \Diamond_i \varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

Let $M = \langle W, R_0, R_1, V \rangle$ be a Kripke model. Then:

- $M, w \models \Box_i \varphi$ iff, for all v, if wR_iv then $M, u \models \varphi$;
- $M, w \models \Diamond_i \varphi$ iff there is v such that wR_iv and $M, u \models \varphi$.

Given a μ-formula φ, define:

$$\Gamma_{\varphi(X)}(A) \to \|\varphi(A)\|^M.$$

Then:

- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$.

Alternation hierarchy

- $\Sigma_0^\mu (= \Pi_0^\mu)$:= set of all formulas with no fixed-point operators.
- Σ_{n+1}^μ is the closure of $\Sigma_n^\mu \cup \Pi_n^\mu$ under:
  - propositional operators;
  - modal operators;
  - μX;
  - and the substitution: if $\varphi(X) \in \Sigma_{n+1}^\mu$ and $\psi \in \Sigma_{n+1}^\mu$ are such that no free variable of ψ becomes bound in $\varphi(\psi)$, then $\varphi(\psi) \in \Sigma_{n+1}^\mu$.
- Π_{n+1}^μ is the dual of Σ_{n+1}^μ.

Game Semantics

We define an evaluation game for $M, w \models \varphi$.

- Two players: Verifier and Refuter.
- Examples of moves:
  - At $\langle \psi \vee \theta, w \rangle$, Verifier moves to one of $\langle \psi, w \rangle$ and $\langle \theta, w \rangle$.
  - At $\langle \Box_i \psi, w \rangle$, Refuter picks v such that $wR_i v$ and moves to $\langle \psi, v \rangle$.
  - At $\langle X, w \rangle$, go to $\langle \mu X.\psi, w \rangle$.
  - At $\langle P, w \rangle$, Verifier wins iff $w \in V(P)$.
- On an infinite run, if the variable with biggest scope which repeats infinitely often is ν, then Verifier wins.

Proposition

Kripke semantics and game semantics are equivalent.

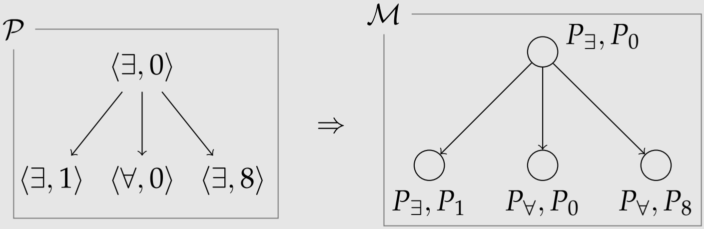
Parity games

- $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega \rangle$
- Two players $\exists$ and $\forall$ move a token in the graph $\langle V_\exists \cup V_\forall, E \rangle$ starting at v_0.
- $\exists$ wins $\rho = v_0, v_1, v_2, \dots$ iff the greatest priority $\Omega(v_i)$ which appears infinitely often in ρ is even.

Proposition

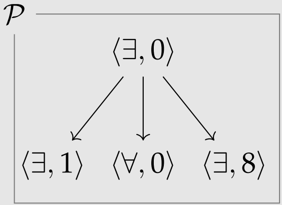
Evaluation games are parity games.

Parity games as unimodal Kripke frames

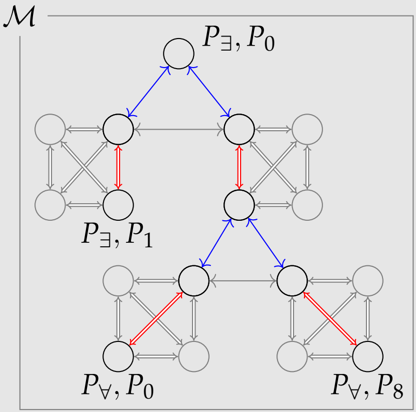


$$W_n := \eta X_n \ldots \mu X_1 \nu X_0 . \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \Diamond X_j) \vee (P_j \wedge P_\forall \wedge \Box X_j)].$$

Parity games as S5 ⊗ S5 frames



$\Rightarrow$



BIMODAL WINNING REGION FORMULAS

$$W'_n := \eta X_n \dots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

BIMODAL WINNING REGION FORMULAS

$$W'_n := \eta X_n \ldots \nu X_0. \bigvee_{0 \leq j \leq n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

Where

- $\blacklozenge \varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \wedge \Diamond_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \wedge \Diamond_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \wedge ((Y \wedge \neg \mathrm{st}) \vee (\varphi \wedge \mathrm{st}))))$; and
- $\blacksquare \varphi := \nu Y.\mathrm{pre}_0 \wedge \mathrm{bd} \to \Box_0(\mathrm{nxt}_0 \wedge \mathrm{pre}_1 \wedge \mathrm{bd} \to \Box_1(\mathrm{nxt}_1 \wedge \mathrm{bd} \to ((Y \wedge \neg \mathrm{st}) \wedge (\varphi \wedge \mathrm{st}))))$,

PROOF SKETCH

- Let n be even. Then $W_n \in \Pi^{\mu}_{n+1}$.
- Suppose that W_n is equivalent to some formula in Π^{μ}_n. Let $\varphi \in \Sigma^{\mu}_n$ be equivalent to $\neg W_n$.
- $f_{\varphi\wedge\varphi}$ takes a pointed model (M, w) to the evaluation game of $M, w \models \varphi \wedge \varphi$ (as a Kripke model).
- Let (M', w') be a fixed-point of $f_{\varphi\wedge\varphi}$. Then

$$\begin{aligned} M', w' \models \neg W_n &\iff M', w' \models \varphi \wedge \varphi \\ &\iff f_{\varphi\wedge\varphi}(M', w') \models W_n \\ &\iff M', w' \models W_n. \end{aligned}$$

- This is a contradiction.

Our result

Theorem

Let F_0, F_1, and F_2 be classes of unimodal Kripke frames closed under isomorphic copies and disjoint unions. If

1. $\circ \leftarrow \circ \rightarrow \circ$ *is a subframe of* F_0 *and* $\circ \rightarrow \circ$ *a subframe of* F_1*; or*
2. $\circ \rightarrow \circ \rightarrow \circ$ *is a subframe of* F_0 *and* $\circ \rightarrow \circ$ *a subframe of* F_1*;*

then the μ-calculus' alternation hierarchy is strict over $\mathsf{F}_0 \otimes \mathsf{F}_1$*. If*

3. $\circ \rightarrow \circ$ *is a subframe of* F_0, F_1, *and* F_2*;*

then the μ-calculus' alternation hierarchy is strict over $\mathsf{F}_0 \otimes \mathsf{F}_1 \otimes \mathsf{F}_2$*.*

Conjecture

Suppose $\circ \rightarrow \circ$ *is a subframe of* F_0 *and* F_1*. We can only show that each μ-formula is equivalent to an alternation-free formula over* $\mathsf{F}_0 \otimes \mathsf{F}_1$*.*

Collapse on multimodal logics

GLP is a provability logic which contains countably many modal operators.

Theorem (Ignatiev [3])

GLP *has the fixed-point property.*

IS5 is an intuitionistic version of S5 which can be treated as a bimodal logic.

Theorem (P. [5])

The μ-calculus collapses to modal logic over IS5*.*

REFERENCES

[1] L. Alberucci, A. Facchini, "The modal μ-calculus hierarchy over restricted classes of transition systems", 2009.

[2] J.C. Bradfield, "Simplifying the modal mu-calculus alternation hierarchy", 1998.

[3] K.N. Ignatiev, "On Strong Provability Predicates and the Associated Modal Logics", 1993.

[4] L. Pacheco, "Exploring the difference hierarchies on μ-calculus and arithmetic—from the point of view of Gale–Stewart games", PhD Thesis, 2023.

[5] L. Pacheco, "Game Semantics for the Constructive μ-Calculus", arXiv:2308.16697, 2023.

[6] L. Pacheco, "The μ-calculus' Alternation Hierarchy is Strict over Non-Trivial Fusion Logics", to appear on Proceedings of FICS 2024.