Intuitionistic Knowledge as a Constructive Diamond

Leonardo Pacheco
Institute of Science Tokyo

(j.w.w. Igor Sedlár) October 28, 2025

Available at: leonardopacheco.xyz/slides/llal9.pdf

Intuitionistic epistemic logic

Artemov and Protopopescu defined a logic IEL to formalize:

Intuitionistic truth implies intuitionistic knowledge.

Our basic logic — IEL_0

IEL_0 consists of

- intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$;

closed under *modus ponens*.

BHK INTERPRETATION

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \rightarrow \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- there is no proof of $\perp$.

Artemov and Protopopescu proposed:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

WHAT IS CONCLUSIVE EVIDENCE?

The examples given by Artemov and Protopopescu are:

- existential generalization,
- zero-knowledge proof,
- testimony of authority,
- classified sources.

Other logics

We obtain the following logics by extending IEL_0:

- $\mathsf{IEL} := \mathsf{IEL}_0 + K\varphi \to \neg\neg\varphi$;
- $\mathsf{IEL}_0^t := \mathsf{IEL}_0 + KK\varphi \to K\varphi$;
- $\mathsf{IEL}^t := \mathsf{IEL} + KK\varphi \to K\varphi = \mathsf{IEL}_0^t + K\varphi \to \neg\neg\varphi$.

Semantics

An IEL_0 model is a tuple $M = \langle W, \preceq, R, V \rangle$ where:

- $\preceq$ is a preorder on W;
- V is monotone w.r.t. $\preceq$;
- wRv implies $w \preceq v$;
- $w \preceq v$ implies, for all u, if vRu then wRu;
- for all w there is v such that wRv.

Define:

- $w \models K\varphi$ iff, for all v, wRv implies $v \models \varphi$.

Proposition

If $w \models \varphi$ and $w \preceq v$, then $v \models \varphi$.

Some Properties

Proposition

Co-reflection implies the following:

- $\mathsf{IEL}_0 \vdash \varphi$ *implies* $\mathsf{IEL} \vdash K\varphi$;
- $\mathsf{IEL}_0 \vdash K\varphi \to KK\varphi$;
- $\mathsf{IEL}_0 \vdash \neg K\varphi \to K\neg K\varphi$.

Constructive Possibility

Definition

$w \models \hat{K}\varphi$ holds iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } vRu \text{ and } u \models \varphi.$$

Proposition

For all IEL *model* M *and world* w*,*

$$M, w \models \hat{K}P \textit{ iff } M, w \models \neg\neg P.$$

Epistemic possibility is impossibility of proof of negation.

Models

A constructive IEL_0-model is a tuple $M = \langle W, w_\perp, \preceq, R, V\rangle$ such that:

- $\preceq$ is a preorder;
- $V(P)$ is closed under $\preceq$ for all $P \in \text{Prop}$;
- if wRv_1 and wRv_2 then there is u such that $v_1, v_2 \preceq u$ and wRu;
- wRv implies $w \preceq v$;
- wRw;
- if $w_\perp \preceq w$ or $w_\perp Rw$, then $w = w_\perp$;
- $w\perp \in V(P)$ for all P.

Definition

$w \models K\varphi$ holds iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } vRu \text{ and } u \models \varphi.$$

BHK Interpretation, again

Definition

$w \models K\varphi$ holds iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } vRu \text{ and } u \models \varphi.$$

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

Completeness - Theories

Definition

A φ-*theory* is a subset Γ^+ of $\mathrm{Sub}(\varphi) \cup \{\bot, \Diamond\bot\}$ such that:

- Γ^+ is closed under IEL-derivations: $X \subseteq \Gamma^+$ and $\mathsf{IEL} \vdash \bigwedge X \to \psi$ implies $\psi \in \Gamma^+$;
- Γ^+ is closed under disjunctions: $\psi \vee \theta \in \Gamma^+$ implies $\psi \in \Gamma^+$ or $\theta \in \Gamma^+$.

Completeness - Segments

Definition

A φ-segment is a pair $\Gamma = \langle \Gamma^+, \Gamma^\Diamond \rangle$ of φ-theories such that:

- $\Gamma^+ \subseteq \Gamma^\Diamond$;
- $K\varphi \in \Gamma^+$ implies $\varphi \in \Gamma^\Diamond$.

Completeness - Canonical Model

Definition

We define a canonical model $M^\varphi := \langle W, \preceq, R, V \rangle$ as follows:

- W is the set of φ-segments;
- $w_\perp$ consists of two copies of the inconsistent φ-theory;
- $\Gamma \preceq \Delta$ iff $\Gamma^+ \subseteq \Delta^+$;
- $\Gamma R \Delta$ iff $\Gamma = \Delta$ or $\Delta^+ = \Gamma^\Diamond$;
- $V(P) = \{\Gamma \in W \mid P \in \Gamma^+\}$.

Lemma

M^φ is a constructive IEL_0-model.

Lemma

For all $\psi \in \mathsf{Sub}(\varphi)$, $M, \Gamma \models \psi$ iff $\psi \in \Gamma$.

Completeness for other logics

- If $K\varphi \to \neg\neg\varphi$ is in the logic, delete the fallible world $w_\perp$.
- If $KK\varphi \to K\varphi$ is in the logic, make R the transitive closure of the defined relation.

BI-MODAL MODELS

We consider bi-relational models

$$\mathcal{W} = \langle W, w_\perp, \preceq, R_K, R_B, V \rangle$$

where

- $\langle W, w_\perp, \preceq, R_K, V \rangle$ is an IEL frame;
- $\langle W, w_\perp, \preceq, R_B, V \rangle$ is an IEL_0 frame;
- $R_K \subseteq R_B$.

These frames are complete for the logic

$$\mathsf{IEL} \oplus \mathsf{IEL}_0 \oplus K\varphi \to B\varphi.$$

Proposition

Belief and knowledge are not interdefinable in this logic.

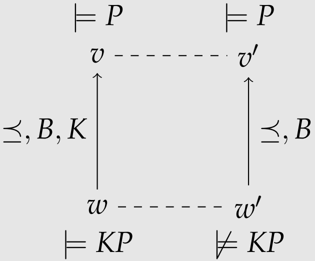
Bisimulations

Fix models $M = \langle W, \preceq, R_K, R_B \rangle$ and $M' = \langle W', \preceq', R'_K, R'_B \rangle$. Let $X \in \{K, B\}$. A X-bisimulation betweeen M and M' is a non-empty relation $Z \subseteq W \times W'$ such that:

1. if wZw' then $w \models P$ iff $w' \models P'$;
2. if wZw' and $w \preceq v$, then there is $v' \in W'$ such that vZv' and $w' \preceq' v'$;
3. if wZw' and $w' \preceq' v'$, then there is $v \in W$ such that vZv' and $w \preceq v$;
4. if wZw' and wR_Xv, then there is $v' \in W'$ such that vZv' and $w'R'_Xv'$;
5. if wZw' and $w'R'_Xv'$, then there is $v \in W$ such that vZv' and wR_Xv.

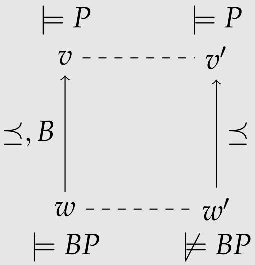
K CANNOT BE DEFINED IN TERMS OF B.

The following two models are B-bisimilar:



B cannot be defined in terms of K

The following two models are K-bisimilar:



Iterative Common Knowledge

Consider a language with multiple knowledge modalities:

$$K_0, K_1, \ldots, K_n.$$

We define:

- $E\varphi := K_0\varphi \wedge K_1\varphi \wedge \cdots \wedge K_n\varphi$; and
- $C\varphi := E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots$

Co-reflection implies that:

$$C\varphi \leftrightarrow E\varphi$$

Thus the *iterative* definition of common knowledge is trivial in this setting.

Consider a language with two knowledge modalities:

$$K_0 \text{ and } K_1.$$

Co-reflection implies:

$$K_0\varphi \to K_1K_0\varphi.$$

Glivenko's Theorem

Theorem (Glivenko)

$\mathsf{CPC} \vdash \varphi$ if and only if $\mathsf{IPC} \vdash \neg\neg\varphi$.

If $\Lambda \in \{\mathsf{IEL}, \mathsf{IEL}_0, \mathsf{IEL}^t, \mathsf{IEL}_0^t\}$, then:

Theorem (Litak, Polzer, Rabenstein)

$\Lambda \vdash \varphi$ if and only if Λ + excluded middle $\vdash \neg\neg\varphi$.

A semantic proof can be obtained as in Chagrov and Zakharyaschev's textbook.

Ignorance in Epistemic Logic

Ignorance *whether* φ holds was defined by Fine as

$$\neg K\varphi \land \neg K\neg\varphi.$$

He also defines and studies other types of ignorance.

Theorem

Co-reflection implies that ignorance is not satisfiable.

PEDANTRY

The following statements are simultaneously true:

- IEL *is* an intuitionistic modal logic.
- IEL *is not* an intuitionistic modal logic.

IEL is intuitionistic

Simpson's six requirements for an intuitionistic modal logic:

- Conservative over IPC.
- Contains all substitution instances of theorems of IPC and is closed under **MP**.
- Adding $\varphi \vee \neg\varphi$ results in a standard classical modal logic.
- Has the disjunction property.
- $\Box$ and $\Diamond$ are independent.
- There is an intuitionistically comprehensible explanation of the meaning of the modalities.

CK AND IK

CK is the modal logic obtained from:

- all intuitionistic tautologies;
- $K_{\Box} := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $K_{\Box} := \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$;
- *modus ponens* and *necessitation*.

IK is obtained by adding to CK the axioms:

- N:= $\neg\Diamond\bot$;
- DP:= $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- FS:= $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

IEL is not intuitionistic — K as a diamond

Theorem

If K is interpreted as a constructive diamond, then its dual satisfies:

$$M, w \models \bar{K}\varphi \text{ iff } M, w \models \varphi.$$

- $\mathrm{N}_\Diamond := \neg K \bot$;
- $\mathrm{DP}_\Diamond := K(\varphi \vee \psi) \to K\varphi \vee K\psi$;
- $\mathrm{FS}_\Diamond := (K\varphi \to \psi) \to (\varphi \to \psi)$.

IEL satisfies $\mathrm{N}_\Diamond$ and $\mathrm{FS}_\Diamond$.
IEL does not satisfy $\mathrm{DP}_\Diamond$.

IEL is not intuitionistic — K as a box

If K is interpreted as a box, then its dual satisfies:

$$M, w \models \hat{K}\varphi \text{ iff } M, w \models \neg\neg\varphi.$$

- $\mathrm{N}_{\Box} := \neg\neg\neg\bot$;
- $\mathrm{DP}_{\Box} := \neg\neg(\varphi \vee \psi) \rightarrow \neg\neg\varphi \vee \neg\neg\psi$;
- $\mathrm{FS}_{\Box} := (\neg\neg\varphi \rightarrow K\psi) \rightarrow K(\varphi \rightarrow \psi)$.

IEL satisfies $\mathrm{N}_{\Box}$ and $\mathrm{FS}_{\Box}$.
IEL does not satisfy $\mathrm{DP}_{\Box}$.

Thank you!

For more on IEL:

- Artemov, Protopopescu, "Intuitionistic epistemic logic".

For more on intuitionistic and constructive modal logics:

- Das, Shillito, de Groot, "Diamond-free parts of intuitionistic modal logics", blog post.