Topological Semantics for IS4

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Available at: leonardopacheco.xyz/slides/llal6.pdf

A QUESTION

A modality is a sequence of $\neg$s, $\Box$s, and $\Diamond$s.

Proposition

S4 *has finitely many modalities (modulo equivalence).*

Question

Does IS4 *have finitely many modalities?*

INTRODUCTION

Theorem

S4 *is complete w.r.t. topological semantics.*

Theorem

IPC *is complete w.r.t. topological semantics.*

We show how to combine the topological semantics for S4 and IPC to obtain:

Theorem (P.)

IS4 *is complete w.r.t. bi-topological semantics.*

This also holds for CS4, S4I, GS4, $GS4^c$.

IS4—a bit of history

- McKinsey–Tarski(50s): topological semantics for S4.
- Fischer Servi(70s): completeness of IS4.
- Simpson(1994): natural deduction for S4.
- Girlando *et al.* (2023): IS4 is decidable.

Existing results

iS4: no diamonds

- de Groot, Shillito (2024)
- Witczak (2019)

IS4

- Davoren (2009): topological semantics for intuitionistic part, relational semantics for modal part.

TOPOLOGICAL SEMANTICS FOR S4

A topological model for IS4 is a tuple $\langle W, \tau, V \rangle$ such that:

- W a non-empty set;
- τ is a topology over W; and
- V is a valuation function.

Define:

- $\|\Box\varphi\|^M = \text{Int}(\|\varphi\|^M)$;
- $\|\Diamond\varphi\|^M = \text{Clo}(\|\varphi\|^M)$.

Valuation for other connectives as usual in classical logics.

Theorem

S4 *is complete w.r.t. topological models.*

Soundness of topological semantics

Proposition

$\Box P \to \Box\Box P$ is valid over topological semantics.

Proof.

$\mathrm{Int}(X) \subseteq \mathrm{Int}(\mathrm{Int}(X))$. $\Box$

Proposition

$\Box P \to P$ is valid over topological semantics.

Proof.

$\mathrm{Int}(X) \subseteq X$. $\Box$

Embedding Kripke models into top. models

- Let $M = \langle W, \sqsubseteq, V \rangle$ be an S4-model.
- Let τ be the topology generated by upsets

$$[w] := \{v \in W \mid w \sqsubseteq v\}.$$

- Define $M^t := \langle W, \tau, V \rangle$.
- For all $w \in W$ and all formula φ,

$$M, w \models \varphi \iff M^t, w \models \varphi.$$

Topological semantics for IPC

A topological model for IPC is a tuple $\langle W, \tau, V \rangle$ such that:

- W is a non-empty set;
- τ is a topology over W;
- V is a valuation function;
- $V(P)$ is an *open set* of τ.

Define:

- $\|P\|^M = V(P)$;
- $\|\bot\|^M = \emptyset$;
- $\|\varphi \wedge \psi\|^M = \|\varphi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M = \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\varphi \rightarrow \psi\|^M = \mathrm{Int}(W \setminus \|\varphi\|^M \cup \|\psi\|^M)$;

Theorem

IPC *is complete w.r.t. topological models.*

IS4 - AXIOMATIZATION

- intuitionistic tautologies;
- $4_\Box := \Box\varphi \to \Box\Box\varphi$;
- $4_\Diamond := \Diamond\Diamond\varphi \to \Diamond\varphi$;
- $T_\Box := \Box\varphi \to \varphi$;
- $T_\Diamond := \varphi \to \Diamond\varphi$;
- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- $N := \neg\Diamond\bot$.

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

IS4 - SEMANTICS

An IS4-model is a tuple $M = \langle W, \preceq, \sqsubseteq, V \rangle$ where:

- W is a non-empty set;
- the *intuitionistic relation* $\preceq$ is a reflexive and transitive relation over W;
- the *modal relation* $\sqsubseteq$ is a reflexive and transitive relation over W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a *valuation function*.

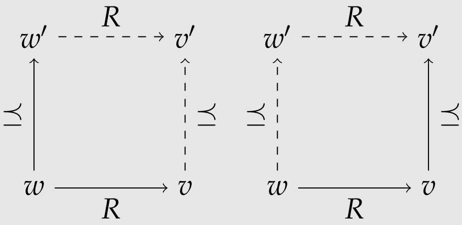
We further require that,

- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$; and
- $\sqsubseteq$ is forward and backward confluent.

Theorem (Fischer Servi)

IS4 *is complete w.r.t.* IS4*-models.*

CONFLUENCES

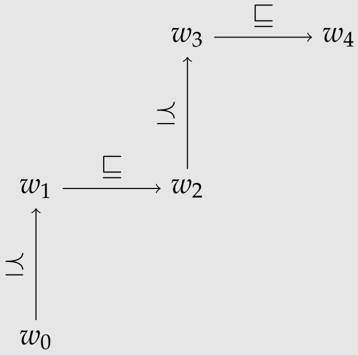


Forward and backward confluence.

Evaluation

- $M, w \models P$ iff $w \in V(P)$;
- $M, w \models \bot$ never holds;
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;
- $M, w \models \varphi \to \psi$ iff, for all $v \in W$, if $w \preceq v$ and $M, v \models \varphi$, then $M, v \models \psi$;
- $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$; and
- $M, w \models \Diamond\varphi$ iff there is u such that wRv and $M, v \models \varphi$.

CONFLUENCE IS NECESSARY



P holds at w_0, w_1, w_2, w_3.
$\Box P \to \Box\Box P$ fails at w_0.

Bi-topological IS4-models

A bi-topological IS4-model is a tuple $M = \langle W, \tau_i, \tau_m, V \rangle$ where:

- W is a non-empty set;
- τ_i and τ_m are topologies over W;
- V assigns to each propositional symbol an i-open set.

We require that:

- the i-interior of any m-open set is m-open; and
- the m-closure of any i-open set is i-open.

VALUATION

We combine the topological valuations for S4 and IPC:

- $\|P\|^M = V(P)$;
- $\|\bot\|^M = \emptyset$;
- $\|\varphi \wedge \psi\|^M = \|\varphi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M = \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\varphi \to \psi\|^M = \mathrm{Int}_i(W \setminus \|\varphi\|^M \cup \|\psi\|^M)$;
- $\|\Box\varphi\|^M = \mathrm{Int}_i(\mathrm{Int}_m(\|\varphi\|^M))$;
- $\|\Diamond\varphi\|^M = \mathrm{Clo}_m(\|\varphi\|^M)$.

SOUNDNESS - I

Lemma

If IS4 *proves* φ, *then* φ *holds over all bi-topological* IS4-*models.*

Straightforward. The key case is:

$$
\begin{aligned}
\|\Box\varphi\| &= \mathrm{Int}_i(\mathrm{Int}_m(\|\varphi\|)) \\
&= \mathrm{Int}_m(\mathrm{Int}_i(\mathrm{Int}_m(\|\varphi\|))) \\
&= \mathrm{Int}_i(\mathrm{Int}_m(\mathrm{Int}_i(\mathrm{Int}_m(\|\varphi\|)))) \\
&= \|\Box\Box\varphi\|
\end{aligned}
$$

(Remmember: the i-interior of any m-open set is m-open.)

EMBEDDING KRIPKE MODELS INTO BI-TOP. MODELS

Let $M = \{W, W^{\perp}, \preceq, \sqsubseteq, V\}$ be an IS4-model. The topologized model M_t is $\langle W, W^{\perp}, \tau_i, \tau_m, V\rangle$, where:

- the basic open sets of τ_i are $[w]_i := \{v \mid w \preceq v\}$; and
- the basic open sets of τ_m are $[w]_m := \{v \mid w \sqsubseteq v\}$.

Proposition

If M is an IS4-model, then M^t is a bi-topological IS4-model.

Proposition

For all $w \in W$ and formula φ,

$$M, w \models \varphi \iff M^t, w \models \varphi.$$

M^t is a bi-topological IS4-model - I

Proposition

The i-interior of any m-open set is m-open.
That is, $\mathrm{Int}_i(U) = \mathrm{Int}_m(\mathrm{Int}_i(U))$*, for all m-open* U*.*

Proof.

- Suppose $w \in \mathrm{Int}_i(U)$.
- Let $w \sqsubseteq w' \preceq w''$.
- By backward confluence, there is v such that $w \preceq v \sqsubseteq w''$.
- As $w \in \mathrm{Int}_i(U)$, $v \in U$.
- $w'' \in U$ since U is m-open.
- So $[w']_i \subseteq U$, and so $w' \in \mathrm{Int}_i(U)$.
- So $[w]_m \subseteq \mathrm{Int}_i(U)$
- We conclude $w \in \mathrm{Int}_m(\mathrm{Int}_i(U))$. $\square$

M^t IS A BI-TOPOLOGICAL IS4-MODEL - II

Proposition

The m-closure of any i-open set is i-open.
That is, $\mathrm{Int}_i(\mathrm{Clo}_m(U)) = \mathrm{Clo}_m(U)$, *for all i-open set U.*

Proof.

- Suppose $w \in \mathrm{Clo}_m(U)$.
- As $w \in \mathrm{Clo}_m(U)$, there is $v \in U$ such that $w \sqsubseteq v$.
- Let $w' \succeq w$.
- By fwd. confluence, there is v' such that $w' \sqsubseteq v'$ and $v \preceq v'$.
- As U is i-open, $v' \in U$.
- Thus $w' \in \mathrm{Clo}_m(U)$.
- We conclude $[w]_i \subseteq \mathrm{Clo}_m(U)$. □

CS4 - Axiomatization

- intuitionistic tautologies;
- $4_\Box := \Box\varphi \to \Box\Box\varphi$;
- $4_\Diamond := \Diamond\Diamond\varphi \to \Diamond\varphi$;
- $T_\Box := \Box\varphi \to \varphi$;
- $T_\Diamond := \varphi \to \Diamond\varphi$.

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

CS4 - Semantics

An CS4-model is a tuple $M = \langle W, W^{\perp}, \preceq, \sqsubseteq, V \rangle$ where:

- $W^{\perp}$ is the set of fallible worlds;
- $\sqsubseteq$ is backwards confluent.
- $M, w \models \Diamond\varphi$ iff for all v if $w \preceq v$ then there is u such that vRu and $M, u \models \varphi$.

Otherwise like an IS4-model.

Theorem (Mendler, de Paiva)

CS4 *is complete w.r.t.* CS4*-models.*

Bi-topological CS4-models

A bi-topological IS4-model is a tuple $M = \langle W, \tau_i, \tau_m, V\rangle$ where:

- W is a non-empty set;
- τ_i and τ_m are topologies over W;
- V assigns to each propositional symbol an *i*-open set.

We require that:

- the *i*-interior of any *m*-open set is *m*-open; and
- the *m*-closure of any *i*-open set is *i*-open.

S4I - Axiomatization

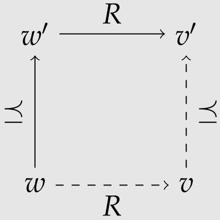
- CS4;
- $CD := \Box(\varphi \vee \psi) \to \Box\varphi \vee \Diamond\psi$;
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$;
- $N := \neg\Diamond\bot$.

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

S4I - Semantics

An S4I-model is a tuple $M = \langle W, \preceq, \sqsubseteq, V \rangle$ where:

- $\sqsubseteq$ is forwards confluent;
- $\sqsubseteq$ is downwards confluent:



- $M, w \models \Box\varphi$ iff for all v suif wRv then $M, v \models \varphi$.

Otherwise like an IS4-model.

Theorem (Balbiani, Diéguez, Fernández-Duque, McLean)

S4I *is complete w.r.t.* S4I-*models.*

BI-TOPOLOGICAL S4I-MODELS

A bi-topological IS4-model is a tuple $M = \langle W, \tau_i, \tau_m, V \rangle$ where:

- W is a non-empty set;
- τ_i and τ_m are topologies over W;
- V assigns to each propositional symbol an *i*-open set.

We require that:

- the *m*-interior of any *i*-open set is *i*-open; and
- the *m*-closure of any *i*-open set is *i*-open.

GS4 AND GS4^c

- $\mathsf{GS4} = \mathsf{IS4} + (\varphi \to \psi) \vee (\psi \to \varphi)$;
- $\mathsf{GS4}^c = \mathsf{GS4} + CD$.

The models are obtained by requiring that $\preceq$ is locally linear: $w \preceq v$ and $w \preceq u$ implies $v \preceq u$ or $w \preceq v$.

Theorem (Balbiani, Diéguez, Fernández-Duque, McLean)

GS4 *and* GS4^c *is complete w.r.t.* GS4- *and* GS4^c*-models, respectively.*

BI-TOPOLOGICAL S4I-MODELS

A bi-topological GS4-model is an IS4 where τ_i is hereditarily extremally disconnected: in every subspace of W, the i-closure of and i-open set is i-open.

Theorem (Bezhanishvili, Bezhanishvili, Lucero-Bryan, van Mill)

τ_i is hereditarily extremally disconnected iff $\preceq$ is locally linear.

CONCLUSION

Theorem (P.)

The following hold:

- IS4 *is complete w.r.t. bi-topological* IS4*-models;*
- CS4 *is complete w.r.t. bi-topological* CS4*-models;*
- S4I *is complete w.r.t. bi-topological* S4I*-models;*
- GS4 *is complete w.r.t. bi-topological* GS4*-models;*
- $\mathsf{GS4}^c$ *is complete w.r.t. bi-topological* $\mathsf{GS4}^c$*-models.*

Future Work

Some ongoing work with David Fernández-Duque and Konstantinos Papafilipou:

- Bi-metric semantics for non-classical variations of S4.
- Transfer the topological and metric results to intuitionistic variations of wK4.