Decidability via fully-labeled non-wellfounded proofs

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Main Results

Decidability for non-classical variations of

- GL
- modal μ-calculus

using fully-labeled non-wellfounded proof systems.

GL

- $\Box(\Box P \to P) \to \Box P$
- complete w.r.t. finite reverse well-founded models

μ-calculus

- add least/greatest (monotone) fixed-points to modal logic
- complete w.r.t. all finite Kripke models

Constructive modal logic

- "Minimal" modal logic based on IPC.
- Models of the form $\langle W, \preceq, R, V \rangle$, where
  - $\preceq$ is the *intuitionistic* relation,
  - R is the *modal* relation.
- Extend the valuation by:
  - $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$;
  - $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ there is u such that vRu and $M, u \models \varphi$.

Other non-classical modal logics

- Intuitionistic modal logics:
  - $(\Diamond P \to \Box Q) \to \Box(P \to Q)$ and $\Diamond(P \vee Q) \to \Diamond P \vee \Diamond Q$,
  - add extra relation between $\preceq$ and R.
- Gödel–Dummett modal logics:
  - $(P \to Q) \vee (Q \to P)$,
  - $\preceq$ is locally linear.

Canonical models in classical modal logic

- Compactness fails for both GL and μ-calculus.
- Therefore the truth lemma fails for them.
- We can overcome this:
  - finite canonical models for GL,
  - final canonical models for μ-calculus (for wK4 models).

Canonical models in intuitionistic modal logic

- Use theories Γ:
  - $\Gamma \vdash_L \varphi$ implies $\varphi \in \Gamma$,
  - $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.
- Works for *completeness*.
- Usually breaks down for decidability:
  - CS4 needs a complicated filtration argument,
  - IS4 needs other methods.

Canonical models in our logics

- How to prove $\mathsf{IGL} \nvdash \varphi$ implies $\mathsf{IGL} \nvdash \Box^n \bot \to \varphi$ for some n?
- Still works if we use an ω-rule:

$$\frac{\{\Box^n \bot \to \varphi\}_{n \in \omega}}{\varphi}.$$

- Not sure if it could work even over variations of $\mathsf{S4}$:
  - Does $\mathsf{CS4}/\mathsf{IS4}$ have finitely many modalities?

Sequents

- In general, intuitionistic proof systems are restricted to sequents of the form:

$$\Gamma \vdash \varphi$$

- This still works even if we only restrict the following rules:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \text{ and } \frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$$

NON-INVERTIBLE RULES

- The rules

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \text{ and } \frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$$

  are *non-invertible*.
- We need more care when doing proof search:
  - we need to choose between different instances of $\to$ and $\Box$,
  - solvable by backtracking or game-theoretic methods.

Labeled Sequents

- Labeled formulas:

$$x : \varphi$$

- Labeled sequents:

$$\mathbf{R}, \Gamma \vdash \Delta,$$

  where **R** is a collection of statements of the form xRy.
- We still need backtracking.
- Not natural for constructive modal logics.

Invertible rules...

$$\vee\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : \varphi \vdash \Delta \qquad \mathbf{R}, \Gamma, x : \psi \vdash \Delta}{\mathbf{R}, \Gamma, x : \varphi \vee \psi \vdash \Delta}$$

$$\vee\mathbf{r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi, x : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \vee \psi}$$

$$\Diamond\mathrm{l}\ \frac{\mathbf{R}, xRy, \Gamma, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \vdash \Delta}\ (y \text{ fresh})$$

$$\Diamond\mathbf{r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A}$$

… and non-invertible rules

$$\to\text{r}\ \frac{\mathbf{R}, \Gamma, x : \varphi \vdash x : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \to \psi}$$

$$\Box\text{r}\ \frac{\mathbf{R}, xRy, \Gamma \vdash y : \varphi}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box\varphi}\ (y \text{ fresh})$$

Fully-labeled Sequents

- Labeled formulas:

$$x : \varphi$$

- Labeled sequents:

$$\mathbf{R}, \Gamma \vdash \Delta,$$

where **R** is a collection of statements of the forms

  - $x \preceq y$,
  - $x' R y'$.

Everyone is invertible

$$\to\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \to \psi}\ (y \text{ fresh})$$

$$\Box\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, yRz, \Gamma \vdash \Delta, z : \varphi}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box\varphi}\ (y, z \text{ fresh})$$

IGL proves $\Box(\Box P \to P) \to \Box P$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{id}\;\dfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P} \qquad \Box\text{l}\;\dfrac{\text{tr}\;\dfrac{\dfrac{\vdots}{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}\;\to\text{l}}{xRy, x : \Box(\Box P \to P) \vdash y : P}\;\Box\text{l}}{x : \Box(\Box P \to P) \vdash x : \Box P}\;\Box\text{r}}{\vdash x : \Box(\Box P \to P) \to \Box P}\;\to\text{r}
$$

NON-WELLFOUNDED PROOFS

- Allow (good) infinite paths in proofs:
  - for GL, require $\{x_i\}_{i\in\omega}$ with $x_i R x_{i+1}$.
  - for μ-calculus, require $\{x_i : \mu^{\alpha_i} X.\varphi\}$ with $\alpha_i > \alpha_{i+1}$.
- Good for completeness.
- Not good for decidability.

IGL proves $\Box(\Box P \to P) \to \Box P$

$$
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\text{id}\,\dfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P}
\qquad
\Box\text{l}\,\dfrac{\text{tr}\,\dfrac{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}
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\dfrac{
\dfrac{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}{xRy, x : \Box(\Box P \to P) \vdash y : P}\,\Box\text{l}
}{x : \Box(\Box P \to P) \vdash x : \Box P}\,\Box\text{r}
}{\vdash x : \Box(\Box P \to P) \to \Box P}\,\to\text{r}
}\,\to\text{l}
$$

$x \mapsto= x$ and $y \mapsto z$.

Soundness

- $\vdash_L \varphi$ and $\not\models_L \varphi$.
- Find a path on the proof of φ which is (stepwise) refuted.
- The path condition will imply a contradiction.

Completeness — non-wellfounded systems

- By a proof(model?)-search argument.
- Keep "saturating" the leaves of the partial proof tree.
- For example,

$$\to\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, x : \varphi \to \psi, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \to \psi}\ (y \text{ fresh})$$

  where there is no z such that $x \preceq z$ is not in the lower sequent and $z : \varphi$ is not on the right of the lower sequent.
- If we ensure every such non-saturated formula is saturated, we get either a proof or a counterexample.

Completeness — non-wellfounded systems

- By a proof-search argument.
- Keep "saturating" the leaves of the partial proof tree, in two alternating phases:
  - only rules which do not introduce statements $x \preceq y$,
  - only rules which introduce statements $x \preceq y$.
- Do some loop-checking to guarantee this search stops:
  - either a good loop to a lower sequent exists,
  - or we can simulate a layer in a lower one.

Main Results

Decidability for non-classical variations of

- GL, and
- modal μ-calculus

using fully-labeled non-wellfounded/cyclic proof systems.

Open: Hilbert-style proof systems.

Non-classical Modal Logic

- Simpson, *"The Proof Theory and Semantics of Intuitionistic Modal Logic"*, 1994.
- Das, Marin, *"On Intuitionistic Diamonds (and Lack Thereof)"*, 2023.
- De Groot *et al.*, *"Semantical Analysis of Intuitionistic Modal Logics"*, 2024. between CK and IK.

GL and μ-calculus

GL

- Das *et al.*, *"Intuitionistic Gödel-Löb Logic, à la Simpson: Labelled Systems and Birelational Semantics"*, 2024.
- Aguilera, P., *"Intuitionistic Gödel-Löb without sharps"*, 2025.

μ-calculus

- Bradfield, Walukiewicz, *"The Mu-Calculus and Model Checking"*, 2018.
- P., *"The Constructive μ-calculus: Game Semantics and Non-Wellfounded Proof System"*, preprint.

Fully labeled sequents

- Marin *et al.*, *"A fully labelled proof system for intuitionistic modal logics"*, 2021
- Girlando *et al.*, *"Intuitionistic S4 Is Decidable"*, 2023.
- (See also Balbiani *et al.*, *"Some Constructive Variants of* S4 *with the Finite Model Property"*, 2021.)
- Girlando *et al.*, *"A simple loopcheck for intuitionistic* K*"*, 2024.