Higher-order feedback computation

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Feedback Turing Machines

Feedback machines have access to information on convergence/divergence of feedback machines.

SOME HISTORY

- Rogers (1967): statements about feedback Turing machines, no proofs.
- Lubarsky (2010): feedback infinite time Turing machines.
- Ackerman, Freer, Lubarsky (2015): feedback Turing machines.
- Aguilera, Lubarsky (2021): feedback hyperjump.

FEEDBACK ORACLE

Feedback Turing machines have access to a halting oracle:

$$h(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^h(n) \text{ converges;} \\ \uparrow, & \text{if } \{e\}^h(n) \text{ diverges;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

When $h(e,n)$ is undefined, the computation $\{e\}^h(n)$ *freezes*.

FREEZING

Let e be such that

$$\{e\}^h(n) := \begin{cases} \text{diverges}, & \text{if } \{n\}^h(n) \text{ converges;} \\ 0, & \text{if } \{n\}^h(n) \text{ diverges.} \end{cases}$$

Then

$$\{e\}^h(e) \text{ converges} \iff \{e\}^h(e) \text{ diverges.}$$

Therefore $\{e\}^h(e)$ freezes.

EXAMPLES

$$\emptyset'(n) := \begin{cases} 1, & \text{if } \{n\}(n) \text{ converges;} \\ 0, & \text{if } \{n\}(n) \text{ diverges.} \end{cases}$$

$$\emptyset''(n) := \begin{cases} 1, & \text{if } \{n\}^{\emptyset'}(n) \text{ converges;} \\ 0, & \text{if } \{n\}^{\emptyset'}(n) \text{ diverges.} \end{cases}$$

$$\emptyset^{(<\omega)}(n) := \begin{cases} 1, & \text{if } \{n\}^{\emptyset^{(i)}}(n) \text{ converges for some } i < \omega; \\ 0, & \text{ otherwise.} \end{cases}$$

Similar constructions can be used to compute the αth Turing jump, for any computable α.

CHARACTERIZATION

Theorem (Ackerman, Freer, Lubarsky)

The following classes coincide:

1. *the feedback semi-computable sets;*
2. *the* Π^1_1 *sets;*
3. *the sets definable by arithmetic inductive operators; and*
4. *the sets of winning positions of Gale–Stewart games whose payoffs are* Σ^0_1.

Second-Order Feedback

2-feedback Turing machines have access to 2 freezing oracles:

$$f_0(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{f_0,f_1}(n) \text{ converges;} \\ \uparrow_0, & \text{if } \{e\}^{f_0,f_1}(n) \text{ diverges;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

$$f_1(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{f_0,f_1}(n) \text{ converges;} \\ \uparrow_0, & \text{if } \{e\}^{f_0,f_1}(n) \text{ diverges;} \\ \uparrow_1, & \text{if } \{e\}^{f_0,f_1}(n) \text{ freezes;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

α-Order Feedback

Fix $\alpha < \omega_1^{\mathrm{ck}}$.
For $\beta < \alpha$, let

$$f_\beta(e,n) := \begin{cases} \downarrow, & \text{if } \{e\}^{\{f_\gamma\}_{\gamma<\alpha}}(n) \text{ converges;} \\ \uparrow_{\beta'}, & \text{if } \{e\}^{\{f_\gamma\}_{\gamma<\alpha}}(n)\ \beta'\text{-freezes } (\beta' < \beta); \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Characterization

Theorem (Aguilera, Lubarsky, P.)

For all $\alpha < \omega_1^{\mathrm{ck}}$, the following classes coincide:

1. *the $(\alpha + 1)$-feedback semi-computable sets;*
2. *the sets definable by $\alpha + 1$ simultaneous arithmetical inductive operators; and*
3. *the sets of winning positions of Gale–Stewart games whose payoffs are differences of α many Σ_2^0 sets.*

SEMI-COMPUTABLE THEN INDUCTIVELY DEFINABLE

Computation history: sequence of states of a Turing machine.

- Converging computation: finite history.
- Diverging computation: infinite history.
- Freezing computation: sequences of finite histories.

Approximate each oracle f_β with an arithmetic inductive definition.

Technical aside

To prove that sets defined by simultaneous arithmetic inductive definitions, we use the μ-arithmetic:

$$t := 0 \mid 1 \mid x \mid t + t \mid t \times t.$$

$$T := X \mid \mu x X.\varphi \mid \nu x X.\varphi,$$

$$\varphi := t = t \mid t \in T \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \bigvee_{i<\omega} \varphi_i \mid \bigwedge_{i<\omega} \varphi_i.$$

INDUCTIVELY DEFINABLE THEN SEMI-COMPUTABLE

Feedback can be used to check quantifiers, conjunctions, and disjunctions. For example:

$$\texttt{forall}(\psi(x), s, i) := \begin{cases} 0, & \text{if } \texttt{eval}(\psi(i), s) = 0 \\ \texttt{forall}(\psi(x), s, i+1), & \text{otherwise} \end{cases}$$

$$\texttt{eval}(\forall x.\psi, s) := \begin{cases} 1, & \text{if } \texttt{forall}(\psi(x), s, 0) \text{ diverges} \\ 0, & \text{otherwise} \end{cases}$$

INDUCTIVELY DEFINABLE THEN SEMI-COMPUTABLE

Higher-order feedback can be used to check fixed-point formulas.

$$\texttt{eval}(t \in \mu x X.\psi, s) := \begin{cases} 1, & \text{if } \texttt{eval}(\psi(t), s[X := \emptyset]) = 1 \\ & \text{or } \texttt{eval}(\psi(t), s[X := \mu x X.\psi]) = 1 \\ \uparrow_\beta, & \text{otherwise} \end{cases}$$

$$\texttt{eval}(t \in \nu x X.\psi, s) := \begin{cases} 1, & \text{if } \texttt{eval}(t \in \mu x X.\neg\psi(\neg X), s)\ \beta\text{-freezes} \\ 0, & \text{otherwise} \end{cases}$$

CHARACTERIZATION

Theorem (Aguilera, Lubarsky, P.)

For all $\alpha < \omega_1^{\mathrm{ck}}$, the following classes coincide:

1. *the $(\alpha + 1)$-feedback semi-computable sets;*
2. *the sets definable by $\alpha + 1$ simultaneous arithmetical inductive operators; and*
3. *the sets of winning positions of Gale–Stewart games whose payoffs are differences of α many Σ_2^0 sets.*

FUTURE WORK

Connection between feedback and reflecting ordinals:

Almost a Theorem

For all $\alpha < \omega_1^{\mathrm{ck}}$, the following classes coincide:

1. *the $(\alpha + 1)$-feedback semi-computable sets, and*
2. *the Σ_1-definable sets in $L_{\beta_{\alpha+1}}$, where $\beta_{\alpha+1}$ is the least $\alpha + 1$-reflecting ordinal.*

There are strict and loose notions of feedback hyperjump.
The following follows from work of Aguilera and Lubarsky:

Theorem

A set of natural numbers is 2-feedback semi-computable iff it is reducible to the loose feedback hyperjump $\mathcal{LO}$.

The relation between higher-order feedback and strict feedback hyperjump is unclear.

REFERENCES

[1] Ackerman, Freer, Lubarsky, "An Introduction to Feedback Turing Computability", 2020.

[2] Aguilera, Lubarsky, Pacheco, "Higher-order feedback computability", 2024.

[3] Rogers Jr., "Theory of Recursive Functions and Effective Computability", 1967.