Higher-order feedback computation

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Feedback Turing machines are Turing machines which can query a halting oracle $h :\subseteq \omega \times \omega \to \{\downarrow, \uparrow\}$, which has information on the convergence or divergence of *feedback* computations. That is, given the code e for a feedback Turing machine and an input n the oracle h answers if the computation $\{e\}^h(n)$ converges or diverges. To avoid a contradiction by diagonalization, feedback Turing machines have two ways of not converging: they can diverge as standard Turing machines, or they can freeze. A feedback Turing machine freezes when it asks the halting oracle h about a pair $\langle e, n \rangle$ not in the domain of h.

Feedback Turing machines were first studied by Ackerman, Freer and Lubarsky [AFL15; AFL20]. They proved that the feedback computable sets are the Δ^1_1 sets and the feedback semi-computable sets are the Π^1_1 sets. We can also show that the feedback semi-computable sets are the winning regions of Gale–Stewart games with Σ^0_1 payoff [Mos09]. It is quite curious that some of the key results of [AFL15] were announced in Rogers' textbook on recursion theory [Rog67], almost 50 years before proofs were published.

A natural question to ask is: what about feedback Turing machines which can ask if computations of the same type converge, diverge, or freeze? These new machines are second-order feedback machines. Note that we must now have a new and stronger notion of freezing to avoid a contradiction by diagonalization. Having defined second-order feedback computation, it is now natural to ask: what about third-, fourth-, and higher-order feedback?

We define αth order feedback Turing machines for each computable ordinal α. We also describe feedback computable and semi-computable sets using inductive definitions and Gale–Stewart games. Specifically, we prove the following level-by-level correspondence:

Theorem 1. *For all $\alpha < \omega_1^{\mathrm{ck}}$, the following classes coincide:*

1. *the $(\alpha + 1)$-feedback semi-computable sets;*
2. *the sets definable by $\alpha + 1$ simultaneous arithmetical inductive operators; and*
3. *the sets of winning positions of Gale–Stewart games whose payoffs are differences of $\alpha + 1$ many Σ^0_2 sets.*

References

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