Epistemic possibility in Artemov and Protopopescu's Intuitionistic Epistemic Logic

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Intuitionistic epistemic logic

Artemov and Protopopescu defined a logic IEL to formalize:

Intuitionistic truth implies intuitionistic knowledge.

INTUITIONISTIC EPISTEMIC LOGIC

Artemov and Protopopescu defined a logic IEL to formalize:

Intuitionistic truth implies intuitionistic knowledge.

IEL consists of

- intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$;
- $T' := K\varphi \to \neg\neg\varphi$;

closed under *modus ponens*.

BHK interpretation

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \to \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- there is no proof of $\bot$.

Artemov and Protopopescu proposed:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

What is conclusive evidence?

The examples given by Artemov and Protopopescu are:

- existential generalization,
- zero-knowledge proof,
- testimony of authority,
- classified sources.

SEMANTICS

An IEL model is a tuple $M = \langle W, \preceq, R, V \rangle$ where:

- $\preceq$ is a preorder on W;
- V is monotone w.r.t. $\preceq$;
- wRv implies $w \preceq v$;
- $w \preceq v$ implies, for all u, if vRu then wRu;
- for all w there is v such that wRv.

Define:

- $w \models K\varphi$ iff, for all v, wRv implies $v \models \varphi$.

Proposition

If $w \models \varphi$ and $w \preceq v$, then $v \models \varphi$.

Some Properties

Proposition

Co-reflection implies the following:

- $\mathsf{IEL} \vdash \varphi$ *implies* $\mathsf{IEL} \vdash K\varphi$;
- $\mathsf{IEL} \vdash K\varphi \to KK\varphi$;
- $\mathsf{IEL} \vdash \neg K\varphi \to K\neg K\varphi$.

Constructive Possibility

Definition

$w \models \hat{K}\varphi$ holds iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } vRu \text{ and } u \models \varphi.$$

Proposition

If $w \models \hat{K}\varphi$ and $w \preceq v$, then $v \models \hat{K}\varphi$.

Possibility implies Double Negation

Proposition

For all IEL *model M and world w, if $w \models \hat{K}P$ then $w \models \neg\neg P$.*

Proof.

We have $\neg\neg\varphi$ iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } v \preceq u \text{ and } u \models \varphi.$$

From $R \subseteq \preceq$, we have $\hat{K}P \to \neg\neg P$. $\square$

DOUBLE NEGATION IMPLIES POSSIBILITY

Proposition

For all IEL *model M and world w, if $w \models \neg\neg P$ then $w \models \hat{K}P$.*

Proof.

By contradiction:

- If $\hat{K}P$ fails at w, there is v such that $w \preceq v$ and, for all v', vRv' implies $v' \not\models P$.
- If $\neg\neg P$ holds at w, there is u such that $v \preceq u$ and $u \models P$.
- uR is not empty; fix $u' \in uR$.
- Since $R \subseteq \preceq$, $u' \models P$.
- As $v \preceq u$, $uR \subseteq vR$.
- Therefore $v \preceq u' \not\models P$. □

BHK interpretation for Possibility

Proposition

For all IEL *model M and world w,*

$$M, w \models \hat{K}P \text{ iff } M, w \models \neg\neg P.$$

Epistemic possibility is impossibility of proof of negation.

Future Work

Alternative semantics where:

- K is interpreted as a constructive diamond;
- strong completeness holds;
- finite model property holds;
- a Glivenko-style theorem holds.

(Ongoing work with Igor Sedlár.)

Thank you!

For more pointers and details, see

- Pacheco, “Epistemic possibility in Artemov and Protopopescu’s intuitionistic epistemic logic”, RIMS Kôkyûroku No.2293, 2024.