Towards a characterization of the μ-calculus' collapse to modal logic

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Fixed-points in modal logic

Provability logic

If X is in the scope of some $\Box$ in $\varphi(X)$, then there is ψ such that

$$\mathsf{GL} \vdash \psi \leftrightarrow \varphi(\psi).$$

Epistemic logic

Common knowledge is defined as

$$C\varphi := \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots$$

where E is the "everyone knows" modality. It can be thought as the greatest fixed-point of the operator

$$X \mapsto EX.$$

THE μ-CALCULUS

The μ-formulas are generated by the grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

Let $M = \langle W, R, V\rangle$ be a Kripke model.
The semantics for μ and ν are as follows:

- $M, w \models \mu X.\varphi$ iff w is in the least fixed point of $\Gamma_{\varphi(X)}$;
- $M, w \models \nu X.\varphi$ iff w is in the greatest fixed point of $\Gamma_{\varphi(X)}$,

where

$$\Gamma_{\varphi(X)}(A) \to \|\varphi(A)\|^M.$$

Alternation depth

The valuation of νX and μY depend on each other:

$$\nu X.\, \underbrace{\mu Y.\, \overbrace{(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)}^{\text{scope of } \mu Y}}_{\text{scope of } \nu X}$$

Alternation depth of φ

Maximum number of codependent alternating μ and ν operators in φ.

Alternation hierarchy

Classifies μ-formulas with respect to their alternation depth.

APPROXIMATING FIXED-POINTS

Consider

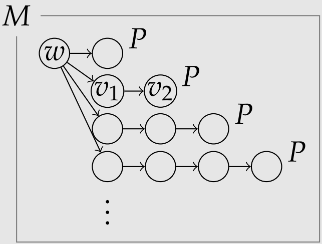
$$\nu X \mu Y.\varphi := \nu X.\mu Y.(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y).$$

To evaluate this formula over $M = \langle W, R, V \rangle$, do as follows:

- Start with $X_0 := W$.
- Y_0 is the least-fixed point of $\Gamma_{\varphi(X_0, Y)}$.
- Set $X_1 := \|\varphi(X_0, Y_0)\|^M$.
- Y_1 is the least-fixed point of $\Gamma_{\varphi(X_1, Y)}$.
- Set $X_2 := \|\varphi(X_1, Y_1)\|$.
- $\dots$
- Repeat until $X_\alpha = X_{\alpha+1}$.
- $\|\nu X \mu Y.\varphi\|^M = X_\alpha$

GAME SEMANTICS — EVALUATION GAMES

Verifier and Refuter discuss whether $\Box\mu X.P \vee \Diamond X$ holds at w.



V : $\Box\mu X.P \vee \Diamond X$ holds at w

R : $\mu X.P \vee \Diamond X$ fails at v_1

V : $P \vee \Diamond X$ holds at v_1

V : $\Diamond X$ holds at v_1

V : X holds at v_2

V : $P \vee \Diamond X$ holds at v_2

V : P holds at v_2

On an infinite run, if the variable with biggest scope which repeats infinitely often is ν, then Verifier wins.

- Key point: on an infinite run, what matters is the *tail*.

GL has the fixed-point property

$$\mathsf{GL} := \mathsf{K} + \Box(\Box P \to P) \to \Box P$$

Theorem (de Jongh, Sambin)

If $\varphi(X)$ is a formula where X is in the scope of some $\Box$, then there is ψ such that

$$\mathsf{GL} \vdash \psi \leftrightarrow \varphi(\psi).$$

S5 does not have the fixed-point property

Theorem (Sacchetti)

Let L *be a logic with the fixed-point property. Then every finite frame for* L *is reverse well-founded.*

Therefore S5 does not have the fixed-point property. However, the μ-calculus collapses to modal logic over S5:

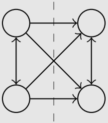
Theorem (Alberucci, Facchini)

Over S5*, every* μ*-formula is equivalent to a formula without fixed-point operators.*

Theorem (P., Tanaka)

The alternation hierarchy collapses to modal logic over S4.3.2.

We may suppose an S4.3.2 frame can be divided into two equivalence classes:



At any long enough game, we will have equivalent positions:

$$\langle \nu X.\varphi, w\rangle \to \cdots \to \langle \Box\psi, v\rangle \to \cdots \to \langle \Box\psi, v'\rangle \to \cdots \to \langle \Box\psi, v''\rangle \to \cdots$$

We can use this fact to show that $\varphi(\varphi(\varphi(\top)) \equiv \varphi(\varphi(\varphi(\varphi(\top))))$.

GENERALIZING THE PROOF

Definition

F is an n-pigeonhole frame iff for all sequence $w_0R^*w_1R^* \cdots R^*w_n$, there is $i < j \leq n$ such that $w_iR = w_jR$.

Definition

The μ-calculus n-uniformly collapses to modal logic over F iff, for all μ-formula φ with X positive,

$$\mu X.\varphi \equiv \varphi^n(\bot) \text{ and } \nu X.\varphi \equiv \varphi^n(\top).$$

Theorem

Fix $n \in \mathbb{N}$. Let F be a class of Kripke frames such that all frames in F are n-pigeonhole frames. Then the μ-calculus $(n+1)$-uniformly collapses to modal logic over F.

Characterizing the collapse

Our theorem does not reverse:

Proposition

Suppose that the μ-calculus $(n+1)$-uniformly collapses to modal logic over F. It does not follow that F is n-pigeonhole.

Proof.

$\mathcal{F}$

$$w_0 \to w_0 \to w_0 \to w_0 \to w_n$$

- By the pigeonhole principle, $\varphi^{n+1}(\bot) \equiv \varphi^{n+2}(\bot)$ over F_{n+1}. Therefore F_{n+1} is $(n+1)$-uniformly collapsing.
- On the other hand, $w_0 R_{n+1} w_1 R_{n+1} \dots R_{n+1} w_n$ witnesses that F_{n+1} is not n-pigeonhole.

$\square$

CHARACTERIZING THE COLLAPSE

We are currently trying to get a good enough reversal:

Question

Let F be a Kripke frame such that the μ-calculus n-uniformly collapses to modal logic over F. Is F is n-pigeonhole?

If the answer is yes, then:

n-uniformly collapse $\Rightarrow$ n-pigeonhole $\Rightarrow$ $(n+1)$-uniformly collapse.

(The answer is yes for $n = 1$ and $n = 2$.)

Common Knowledge

- Common knowledge is defined by

$$\begin{aligned} C\varphi &:= \varphi \wedge E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots \\ &\equiv \mu X.\varphi \wedge EX. \end{aligned}$$

  where E is the "everyone knows" modality.
- If there are two or more agents, common knowledge is not equivalent to a modal formula.
- The μ-calculus does not collapse if we have two or more agents:

Theorem

The μ-calculus' alternation hierarchy is strict over $\mathsf{S5}_2$ frames.

Parity games

- Two players $\exists$ and $\forall$ move a token in a graph.
- Each vertex is labeled with a natural number and an owner.
- $\exists$ wins a run $\rho = v_0, v_1, v_2, \ldots$ iff the greatest label which appears infinitely often in ρ is even.
- Key point: on an infinite run, what matters is the *tail*.
- Evaluation games for the μ-calculus are parity games.

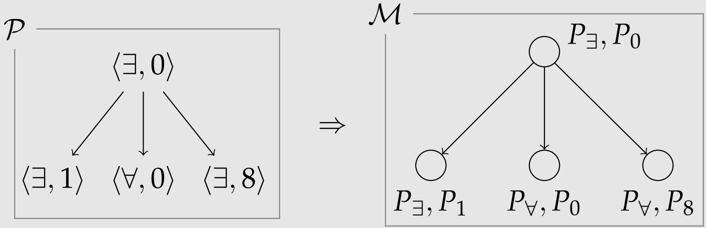
PARITY GAMES AS KRIPKE MODELS

W_n describes the winning region for $\exists$ in parity games where n is the maximum parity:

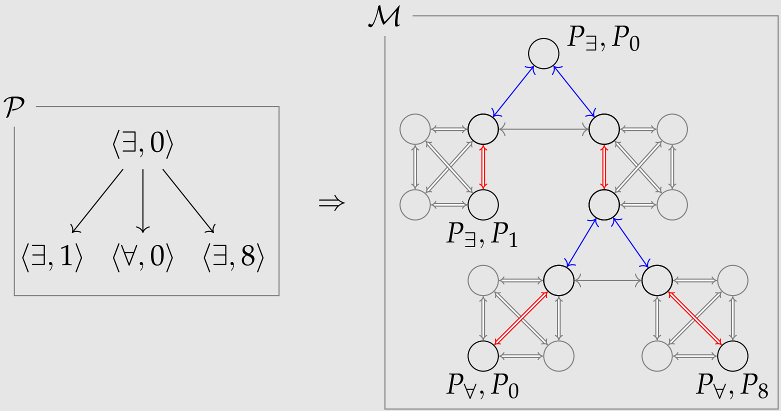
$$W_n := \eta X_n \ldots \nu X_0 . \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \Diamond X_j) \vee (P_j \wedge P_\forall \wedge \Box X_j)].$$

Theorem (Bradfield)

Let $n \in \omega$, then W_n is not equivalent to any formula with less alternation.



Parity games as $\mathsf{S5}_2$ models



Bimodal winning region formulas

$$W'_n := \eta X_n \dots \nu X_0. \bigvee_{0 \le j \le n} [(P_j \wedge P_\exists \wedge \blacklozenge X_j) \vee (P_j \wedge P_\forall \wedge \blacksquare X_j)].$$

Where

- $\blacklozenge \varphi := \mu Y.\text{pre}_0 \wedge \text{bd} \wedge \Diamond_0(\text{nxt}_0 \wedge \text{pre}_1 \wedge \text{bd} \wedge \Diamond_1(\text{nxt}_1 \wedge \text{bd} \wedge ((Y \wedge \neg\text{st}) \vee (\varphi \wedge \text{st}))))$; and
- $\blacksquare \varphi := \nu Y.\text{pre}_0 \wedge \text{bd} \to \Box_0(\text{nxt}_0 \wedge \text{pre}_1 \wedge \text{bd} \to \Box_1(\text{nxt}_1 \wedge \text{bd} \to ((Y \wedge \neg\text{st}) \wedge (\varphi \wedge \text{st}))))$,

GENERALIZING THE NON-COLLAPSE OVER FUSIONS

The strictness over $\mathsf{S5}_2$ can be generalized to:

Theorem

The μ-calculus' alternation hierarchy is strict over interesting fusions of modal logics.

AN OPEN PROBLEM

When does the μ-calculus' alternation hierarchy collapse over an *interesting* multimodal logic?

Example (Ignatiev)

The fixed-point theorem holds over GLP.

Example (P.)

The μ-calculus collapses to modal logic over MIPQ *(a.k.a.* IS5*).*

Non-example

The μ-calculus collapses to modal logic over epistemic logic with knowledge and belief for only one agent.

THANK YOU!

- The μ-calculus $(n+1)$-uniformly collapses to modal logic over n-pigeonhole frames.
- Are n-uniformly collapsing frames also n-pigeonhole?
- The μ-calculus' alternation hierarchy is strict over most multimodal settings.
- Which restriction do we need to add between the modalities for the μ-calculus to collapse?

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