A non-wellfounded proof system for the constructive μ-calculus

Leonardo Pacheco
Institute of Science Tokyo

27 March 2026

Available at: leonardopacheco.xyz/slides/260327sciencetokyo.pdf

MAIN RESULTS

Completeness for fully-labeled non-wellfounded proof systems for non-classical variations of the modal μ-calculus:

Theorem

- $\mathsf{lab}\mu\mathsf{CK}_\omega \vdash \varphi$ *iff* $\mathsf{CK} \models \varphi$;
- $\mathsf{lab}\mu\mathsf{IK}_\omega \vdash \varphi$ *iff* $\mathsf{IK} \models \varphi$;
- $\mathsf{lab}\mu\mathsf{GK}_\omega \vdash \varphi$ *iff* $\mathsf{GK} \models \varphi$.

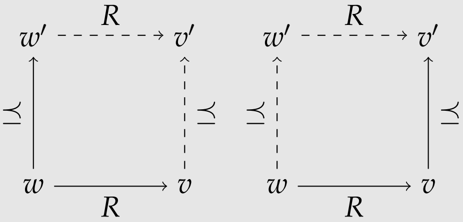
We make essential use of game semantics for the constructive μ-calculus.

Constructive modal logic

- "Minimal" modal logic based on IPC.
- Models of the form $\langle W, W^{\perp}, \preceq, R, V \rangle$, where
  - $\preceq$ is the *intuitionistic* relation,
  - R is the *modal* relation.
- Extend the valuation by:
  - $M, w \models \Box\varphi$ iff, for all $v, u \in W$, if $w \preceq v$ and vRu, then $M, u \models \varphi$;
  - $M, w \models \Diamond\varphi$ iff, for all $v \in W$, if $w \preceq v$ there is u such that vRu and $M, u \models \varphi$.

INTUITIONISTIC AND GÖDEL–DUMMETT MODAL LOGICS

For intuitionistic modal logics, require:



For Gödel–Dummet modal logics, further require:

- if $w \preceq v$ and $w \preceq u$, then $v \preceq u$ or $u \preceq v$.

CONSTRUCTIVE μ-CALCULUS

Add least and greatest fixed-points:

- $\|\mu X.\varphi(X)\| = \mathrm{LFP}(\mathrm{X} \mapsto \|\varphi(\mathrm{X})\|)$;
- $\|\nu X.\varphi(X)\| = \mathrm{GFP}(\mathrm{X} \mapsto \|\varphi(\mathrm{X})\|)$;

Outline of $\mathcal{G}(M, w \models \varphi)$

Two players:

- I and II.

Two roles:

- V and R.

Positions of the form:

- $\langle w, \varphi, \mathsf{Q} \rangle$.

Discuss wether:

- $M, w \models \varphi$.

RULES FOR V

Position	Admissible moves
$\langle v, P, \mathsf{Q}\rangle$ and $v \notin V(P)$	$\emptyset$
$\langle v, \bot, \mathsf{Q}\rangle$ and $v \notin W^{\bot}$	$\emptyset$
$\langle v, \psi \vee \theta, \mathsf{Q}\rangle$	$\{\langle v, \psi, \mathsf{Q}\rangle, \langle v, \theta, \mathsf{Q}\rangle\}$
$\langle v, \psi ? \theta, \mathsf{Q}\rangle$	$\{\langle v, \psi, \bar{\mathsf{Q}}\rangle, \langle v, \theta, \mathsf{Q}\rangle\}$
$\langle v, \hat{\Diamond}\psi, \mathsf{Q}\rangle$	$\{\langle u, \psi, \mathsf{Q}\rangle \mid vRu\}$
$\langle v, \mu X.\psi_X, \mathsf{Q}\rangle$	$\{\langle v, \psi_X, \mathsf{Q}\rangle\}$
$\langle v, X, \mathsf{Q}\rangle$	$\{\langle v, \mu X.\psi_X, \mathsf{Q}\rangle\}$

RULES FOR R

Position	Admissible moves
$\langle v, P, \mathsf{Q}\rangle$ and $v \in V(P)$	$\emptyset$
$\langle v, \bot, \mathsf{Q}\rangle$ and $v \in W^{\bot}$	$\emptyset$
$\langle v, \psi \wedge \theta, \mathsf{Q}\rangle$	$\{\langle v, \psi, \mathsf{Q}\rangle, \langle v, \theta, \mathsf{Q}\rangle\}$
$\langle v, \psi \to \theta, \mathsf{Q}\rangle$	$\{\langle u, \psi ? \theta, \mathsf{Q}\rangle \mid v \preceq v\}$
$\langle v, \Box\psi, \mathsf{Q}\rangle$	$\{\langle u, \psi, \mathsf{Q}\rangle \mid v \preceq ; Ru\}$
$\langle v, \Diamond\psi, \mathsf{Q}\rangle$	$\{\langle u, \hat{\Diamond}\psi, \mathsf{Q}\rangle \mid v \preceq u\}$
$\langle v, \nu X.\psi_X, \mathsf{Q}\rangle$	$\{\langle v, \psi_X\rangle, \mathsf{Q}\}$
$\langle v, X, \mathsf{Q}\rangle$	$\{\langle v, \nu X.\psi_X, \mathsf{Q}\rangle\}$

Equivalence

The birelational Kripke semantics and game semantics for CK are equivalent.

Theorem

If $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ is a CK*-model, $w \in W$ and φ is a well-named μ-sentence, then:*

- I *has a winning strategy for $\mathcal{G}(M, w \models \varphi)$ if and only if $M, w \models \varphi$; and*
- II *has a winning strategy for $\mathcal{G}(M, w \models \varphi)$ if and only if $M, w \not\models \varphi$.*

FULLY-LABELED SEQUENTS

- Labeled formulas:

$$x : \varphi$$

- Labeled sequents:

$$\mathbf{R}, \Gamma \vdash \Delta,$$

  where **R** is a collection of statements of the forms
  - $x \preceq y$,
  - $x'Ry'$.

Some rules — I

$$\wedge l\ \frac{\mathbf{R}, \Gamma, x : \varphi \wedge \psi, x : \varphi, x : \psi \vdash \Delta}{\mathbf{R}, \Gamma, x : \varphi \wedge \psi, \vdash \Delta}$$

$$\wedge r\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \wedge \psi, x : \varphi \qquad \mathbf{R}, \Gamma \vdash \Delta, x : \varphi \wedge \psi, x : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \wedge \psi}$$

$$\Diamond l\ \frac{\mathbf{R}, xRy, \Gamma, x : \Diamond\varphi, y : \varphi \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond\varphi \vdash \Delta}\ (y \text{ fresh})$$

$$\Box r\ \frac{\mathbf{R}, x \preceq y, yRz, \Gamma \vdash \Delta, x : \Box\varphi, y : \varphi}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box\varphi}\ (y, z \text{ fresh})$$

$$\rightarrow r\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, x : \varphi \rightarrow \psi, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \rightarrow \psi}\ (y \text{ fresh})$$

Some Rules — II

$$\eta\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : \eta X.\varphi_X, \boxed{x : \varphi_X} \vdash \Delta}{\mathbf{R}, \Gamma, \boxed{x : \eta X.\varphi_X} \vdash \Delta}$$

$$\eta\mathbf{r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \eta X.\varphi_X, \boxed{x : \varphi_X}}{\mathbf{R}, \Gamma \vdash \Delta, \boxed{x : \eta X.\varphi}}$$

$$\text{regen-l}\ \frac{\mathbf{R}, \Gamma, x : X, \boxed{x : \varphi_X} \vdash \Delta}{\mathbf{R}, \Gamma, \boxed{x : X} \vdash \Delta}$$

$$\text{regen-r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : X, \boxed{x : \varphi_X}}{\mathbf{R}, \Gamma \vdash \Delta, \boxed{x : X}}$$

A Simple Example — $\eta X.\Box X$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x \preceq x', x'Ry, y \preceq y', y'Rz \vdash z : X}}{x \preceq x', x'Ry \vdash y : \Box X}}{x \preceq x', x'Ry \vdash y : X}}{\vdash x : \Box X}}{\vdash x : \eta X.\Box X}
$$

Soundness — Refutations

Let:

- φ be a sentence,
- $S = \mathbf{R}, \Gamma \vdash \Delta$ be a sequent where all formulas are subformulas of φ,
- $M = \langle W, W^{\perp}, \preceq, R, V \rangle$ be a CK-model, and
- τ be a strategy for II on $\mathcal{G}(M, w \models \varphi)$.

A function $r : \ell(S) \to W$ is a *τ-refutation* of S on M iff

- for all $x, y \in \ell(S)$, $xRy \in \mathbf{R}$ implies $r(x)Rr(y)$ and $x \preceq y \in \mathbf{R}$ implies $r(x) \preceq r(y)$;
- if $x : \psi \in \Gamma$, then τ is a winning strategy for II at $\langle r(x), \psi, \mathsf{R} \rangle$; and
- if $x : \psi \in \Delta$, then τ is a winning strategy for II at $\langle r(x), \psi, \mathsf{V} \rangle$.

Soundness — Proof Sketch

- Suppose T is a proof if φ and $M, w \not\models \varphi$.
- There is τ winning for II in $\mathcal{G}(M, w \not\models \varphi)$.
- The root S_0 of T is τ-refuted by $r_0 : x \mapsto w$.
- If S_i is τ-refuted by r_i, there is refutation r_{i+1} and a premise S_{i+1} such that r_{i+1} is a τ-refutation of S_{i+1}.
- The trace conditions of games and proofs contradict.

Completeness

- By a proof(model?)-search argument.
- Keep "saturating" the leaves of the partial proof tree.
- For example,

$$\rightarrow\mathrm{r}\ \frac{\mathbf{R}, x \preceq y, \Gamma, y : \varphi \vdash \Delta, x : \varphi \rightarrow \psi, y : \psi}{\mathbf{R}, \Gamma \vdash \Delta, x : \varphi \rightarrow \psi}\ (y \text{ fresh})$$

  where there is no z such that $x \preceq z$ is not in the lower sequent and $z : \varphi$ is not on the right of the lower sequent.
- If we ensure every such non-saturated formula is saturated, we get either a proof or a counterexample.

Main Results and Ongoing Work

Theorem

- $\mathsf{lab}\mu\mathsf{CK}_\omega \vdash \varphi$ *iff* $\mathsf{CK} \models \varphi$;
- $\mathsf{lab}\mu\mathsf{IK}_\omega \vdash \varphi$ *iff* $\mathsf{IK} \models \varphi$;
- $\mathsf{lab}\mu\mathsf{GK}_\omega \vdash \varphi$ *iff* $\mathsf{GK} \models \varphi$.

Ongoing work: decidability via cyclic proof systems.

References

- Simpson, *"The Proof Theory and Semantics of Intuitionistic Modal Logic"*, 1994.
- Das, Marin, *"On Intuitionistic Diamonds (and Lack Thereof)"*, 2023.
- De Groot *et al.*, *"Semantical Analysis of Intuitionistic Modal Logics between CK and IK"*, 2024.
- Marin *et al.*, *"A fully labelled proof system for intuitionistic modal logics"*, 2021
- P., *"The Constructive μ-calculus: Game Semantics and Non-Wellfounded Proof System"*, preprint.