On the weak hierarchy of μ-calculus

A Thesis presented by

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Contents

Chapter 1

Introduction

In this thesis we present the weak μ-calculus and weak μ-arithmetic and their alternations hierarchies. We also prove a refinement of a result in Reverse Mathematics related to the μ-arithmetic and the determinacy of the finite levels of the difference hierarchy of Σ^0_2.

On chapter 2 we explain the basic concepts used in this thesis and some results in the literature. The main concepts are the μ-calculus and the μ-arithmetic. The μ-calculus is an extension of modal logic via fixed points. Its main point is the ability to write formulas expressing the idea of "eventually" and "always". For example $\mu X.\Diamond X \vee P$ means "eventually P holds" and $\nu X.\Box X \wedge P$ is means "P always holds". The μ-calculus has two operators μ and ν used to express fixed points. By alternating these two types of operators, we obtain more complex formulas and we can define hierarchies counting these alternations. We prove that the alternation hierarchies are strict. The μ-arithmetic is obtained by adding these fixpoints to first-order arithmetic and also has a similarly defined alternation hierarchy with the same properties as the alternation hierarchy for the μ-calculus. These systems are deeply connected. We also consider transfinite versions of these systems by adding recusive conjunction and disjunction of formulas. In the second half of this section, we give a short overview of Gale-Steward games, determinacy, and the difference hierarchy for Σ^0_2. Furthermore we show the connection between the μ-arithmetic and the difference hierarchy for Σ^0_2. At the end of chapter 2, we shortly introduce Reverse Mathematics and present some of its connections with determinacy.

On chapter 3 we define the weak μ-calculus and the weak μ-arithmetic. These are weaker versions of the systems defined in Chapter 2 restricting the interation between μ and ν operators. The sets definable in these weak systems are all definable by formulas in a low level of the alternation hierarchy for the full systems. At last, we show the connection between the weak systems and the difference hierarchy for Σ^0_1.

On chapter 4 we refine a result of Möllerfeld and Heinatsch. This result states that a formalized version of the μ-arithmetic inside second-order arithmetic is equivalent to the determinacy of games with payoff in the finite levels of the difference hierarchy for $\mathbf{\Sigma}^0_2$. Here we define subsystems of the μ-arithmetic limiting the ability to apply the fixed point operators and show that each one of these subsystems is equivalent to determinacy for some level of the determinacy hierarchy for $\mathbf{\Sigma}^0_2$.

Chapters 3 and 4 are not directly related, but the proofs in them share some ideas. Also, at the end of chapters 3 and 4 we present some questions we plan to consider on future work.

Chapter 2

Preliminaries

2.1 Modal logic

The modal logic is an extention of the propositional logic. Its main feature is the ability to express "necessity" and "possibility".

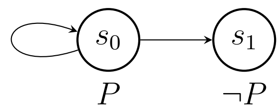
Fix a set of propositional symbols $Prop$ and a set of propositional variables Var. We define the formulas of modal logic by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi.$$

Here P and $\neg P$ are a propositional symbol and the negation of a propositional symbol, respectively. X is a propositional variable. Let φ and ψ be formulas of modal logic, then $\varphi(X\backslash\psi)$ is defined to be φ with all occurences of X substituted by ψ. Note that, by this definition, if X does not appear in φ then $\varphi(X\backslash\psi)$ is just φ. $\wedge$ and $\vee$ have their usual meaning from propositional logic. The modal operators $\Box$ and $\Diamond$ denote necessity and possibility respectively, we will explain their meaning after defining the models of modal logic.

Our models are labeled transition systems of the form $S = (S, E, \rho)$ where S is the set of states, $E \subseteq S \times S$ are the transitions and $\rho : Prop \to \mathcal{P}(S)$ assigns to each proposition P the states in which P is valid. Each state of our transition symbols can be though as a valuation of propositional logic. We define a valuation to be a function $V : Var \to \mathcal{P}(S)$ assingning to each propositional variable X the states where it holds.

Example 1. Let $S = \{s_0, s_1\}$,$E = \{\langle s_0, s_0\rangle, \langle s_0, s_1\rangle\}$ and $\rho(P) = \{s_0\}$. We can draw this transition system as:



Given a transition system S, a valuation V and a modal formula φ, $\|\varphi\|_V^S$ denotes the states $s \in S$ where φ holds. The meaning for $\wedge, \vee$ and $\neg$ are the same as the meaning in propositional logic. $s \in \|\Box\varphi\|_V^S$ means that $t \in \|\varphi\|_V^S$ for all states $t \in S$ which are accessible from s (i.e., $\langle s, t\rangle \in E$). $s \in \|\Diamond\varphi\|_V^S$ means that $t \in \|\varphi\|_V^S$ for some state $t \in S$ which is accessible from s. Formally, we have:

Definition 1 (Kripke Semantics)**.** Given a transition system S and a valuation $V : Var \to \mathcal{P}(S)$, we define

$$\begin{aligned}
\|P\|_V^S &= \rho(P) \\
\|X\|_V^S &= V(X) \\
\|\neg\varphi\|_V^S &= S \setminus \|\varphi\|_V^S \\
\|\varphi \wedge \psi\|_V^S &= \|\varphi\|_V^S \cap \|\psi\|_V^S \\
\|\varphi \vee \psi\|_V^S &= \|\varphi\|_V^S \cup \|\psi\|_V^S \\
\|\Box\varphi\|_V^S &= \{s | \forall t \in S. \langle s, t\rangle \in E \implies t \in \|\varphi\|_V^S\} \\
\|\Diamond\varphi\|_V^S &= \{s | \exists t \in S. \langle s, t\rangle \in E \wedge t \in \|\varphi\|_V^S\}
\end{aligned}$$

Note that $\Box$ and $\Diamond$ are dual, i.e., for all formulas $\neg\Box\varphi$ is equal to $\Diamond\neg\varphi$. If the context permits, we omit S and V in the notation above.

As we can see from the definition above, the meaning of necessity and possibility can change quite a lot depending on the transition system under consideration. Indeed, modal logic can also be used to talk, for example, about time, epistemology(knowledge) and deontology(obligation). For more information about this and modal logic in general, see [1]. Also note that we can extend the modal logic to include more modal operators.

Before ending this section, we expand Example 1.

Example 2. Consider the transition system S of Example 1, then:

- $s_0 \in \|P\|$ and $s_0 \in \|\Diamond P\|$, but $s_1 \notin \|P\|$ and $s_1 \notin \|\Diamond P\|$.

- $s_1 \in \|\Box P\|$, but $s_0 \notin \|\Box P\|$.
- For all modal formulas φ, $s_1 \in \|\Box\varphi\|$.
- For all modal formulas φ, $s_1 \notin \|\Diamond\varphi\|$.

2.2 μ-calculus

In this section, we define the μ-calculus and explain some of its basic properties. The μ-calculus is an extension of modal logic and was first defined by Kozen in [14].

The modal μ-calculus is an extension of the (propositional) modal logic by fixpoints. We assume (fixed) countable sets of propositions and variables. We use $P, Q, R, \ldots$ as propositional symbols and $X, Y, Z, \ldots$ for variable symbols. The formulas of μ-calculus are defined by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

We can introduce negation on the μ-calculus by defining it to follow the usual rules on connectives and modalities of modal logic and follow

$$\neg\mu X.\varphi = \nu X.\neg\varphi[X \mapsto \neg X].$$

We call these formulas μ-formulas. We can suppose there is no repetition between the bounded variables in a formula, as we can always exchange the variables by new ones. For example $\nu X.(X \vee \mu X.(P \wedge X))$ can be rewritten as $\nu Y.(Y \vee \mu X.(P \wedge X))$. Let φ be a μ-formula, then we say the μ-operator in $\mu X.\varphi$ binds X in φ and that every instance of X is bound in $\mu X.\varphi$. If X is bound by a μ-operator, we call X a μ-variable, and if X is bound by a ν-operator, we call X a ν-variable. If a instace of X in φ is not bound, we say it is free. If a formula φ has no free variables, we say it is closed.

We also consider a transfinite extension of the μ-calculus by adding the rule that if φ_n is a recursive enumeration of μ-formulas, then $\bigvee_{n\in\omega}\varphi_n$ is also a μ-formula. In this thesis, we only use $\bigvee$ to mean recursive conjunction, as arbitrary disjunction does not make sense in our context, as it would allow for arbitrary sets to be definable via transfinite μ-formulas. The reason for this will be clear after we give the semantics of the μ-calculus.

Before defining semantics for the μ-calculus, we discuss the intended meaning for μ and ν. μ is used to indicate "liveliness properties" and ν

is used to indicate "safety properties". For example $\mu X.\Diamond X \vee P$ is intended to mean "eventually P holds" and $\nu X.\Box X \wedge P$ is intended to mean "P always holds".

We now discuss the semantics for the μ-calculus. We follow [20]. We consider truth over a labeled transition system $\mathcal{S} = (S, E, \rho)$ where S is the set of states, $E \subseteq S \times S$ are the transitions and $\rho : Prop \to \mathcal{P}(S)$ assigns to each proposition P the states in which P is valid.

As the μ-calculus is an extension of the modal logic, we give $\neg, \wedge$ and $\Box$ their usual meanings. If $\varphi(X)$ is a μ-formula with X as a free variable and $\mathcal{S}$ is a transition system, given the modal logic semantics, we have that $\|\varphi(U)\|$ is a subset of S for each $U \subseteq S$. The formula $\mu X.\varphi(X)$ indicates the least fixed point of the function $\Gamma : \mathcal{P}(S) \to \mathcal{P}(S)$ defined by $\Gamma(U) = \|\varphi(U)\|^{\mathcal{S}}$. Note that as Γ is monotone, such a least fixed point exists. For $\bigvee$ we just take its meaning as union, just like $\wedge$.

We define formally the semantics of the μ-calculus as follows:

Definition 2 (Extentional Semantics)**.** Given a transition system $\mathcal{S}$ and a valuation $V : Var \to \mathcal{P}(S)$, we define

$$\begin{aligned}
\|P\|_V^{\mathcal{S}} &= \rho(P)\\
\|X\|_V^{\mathcal{S}} &= V(X)\\
\|\neg\varphi\|_V^{\mathcal{S}} &= S \setminus \|\varphi\|_V^{\mathcal{S}}\\
\|\varphi \wedge \psi\|_V^{\mathcal{S}} &= \|\varphi\|_V^{\mathcal{S}} \cap \|\psi\|_V^{\mathcal{S}}\\
\|\Box\varphi\|_V^{\mathcal{S}} &= \{s | \forall t \in S. \langle s, t\rangle \in E \implies t \in \|\varphi\|_V^{\mathcal{S}}\}\\
\|\mu X.\varphi\|_V^{\mathcal{S}} &= \bigcap\{U \subseteq S | \|\varphi\|_{V[X\to U]}^{\mathcal{S}} \subseteq U\}\\
\|\bigvee_{n\in\omega} \varphi_n\|_V^{\mathcal{S}} &= \bigcup_{n\in\omega} \|\varphi_n\|_V^{\mathcal{S}}
\end{aligned}$$

in the above definition, $V[Z \to U](X) = U$ if $X = Z$ and $V[Z \to U](X) = V(X)$ otherwise. We also define $\|\varphi \vee \psi\|_V^{\mathcal{S}}, \|\Diamond\varphi\|_V^{\mathcal{S}}$ and $\|\nu X.\varphi\|_V^{\mathcal{S}}$ using the already defined relations. If the context permits, we omit $\mathcal{S}$ and V in the notation above.

We also define another semantic for the μ-calculus. Given a μ-formula φ, a transition system $\mathcal{S}$ and a valuation V, we define a game to decide if $s \in \mathcal{S}$ satisfies φ. We denote this game by $\mathcal{G}_V^{\mathcal{S}}(s, \varphi)$. It has two players, V(erifier) and R(efuter). As their name says V wants to show that s satisfies

φ and R wants to show that s does not satisfy φ. The game is played over a directed graph where each vertex is owned by one of the players, who decides the next vertice to be visited. The game vertices are pairs composed of a subformula of φ and a state $s \in S$. We give the ownership of each the game states according to the subformula in it. For example, if the game vertice is $\langle \psi_1 \wedge \psi_2, s \rangle$, the R chooses either $\langle \psi_1, s \rangle$ or $\langle \psi_2, s \rangle$ and he tries to choose one which is going to be false. If the game vertice is $\langle \Diamond\psi, s \rangle$, V tries to choose a state t accessible from s such that $\langle \psi, t \rangle$ is valid. Furthermore, if a play is infinite, R wants to play so that, among the variables which were passed through infinitely many times, the outermost variable is a μ-variable. Formally, we have:

Definition 3 (Game Semantics)**.** Given a transition system S, a state $s_0 \in S$, a valuation $V : Var \to \mathcal{P}(S)$ and a μ-calculus formula φ we define the game $\mathcal{G}_V^S(s_0, \varphi)$: The game vertices are the pairs $\langle s, \psi \rangle$ where $s \in S$ and ψ is a subformula of φ. The initial state is $\langle s_0, \varphi \rangle$.

For the game edges we have:

- If $\psi_0 \wedge \psi_1$ is a subformula of φ then $\langle s, \psi_0 \wedge \psi_1 \rangle \to \langle s, \psi_0 \rangle$ and $\langle s, \psi_0 \wedge \psi_1 \rangle \to \langle s, \psi_1 \rangle$ are edges.
- If $\Box\psi$ is a subformula of φ and $\langle s, t \rangle \in E$, then $\langle s, \Box\psi \rangle \to \langle t, \psi \rangle$ is an edge. The same holds if we substitute $\Box$ by $\Diamond$.
- If $\mu X.\psi$ is a subformula of φ then $\langle s, \mu X.\psi \rangle \to \langle s, \psi \rangle$ and $\langle s, X \rangle \to \langle s, \mu X.\psi \rangle$ are edges. Again, we do it similarly for $\nu X.\psi$

Furthermore we define the ownership of the vertices in the following way: V owns $\langle s, \psi_0 \vee \psi_1 \rangle$, $\langle s, \Diamond\psi \rangle$, $\langle s, P \rangle$ if $s \notin \rho(P)$ and $\langle s, Z \rangle$ if $s \notin V(Z)$. R owns $\langle s, \psi_0 \wedge \psi_1 \rangle$, $\langle s, \Box\psi \rangle$, $\langle s, P \rangle$ if $s \in \rho(P)$ and $\langle s, Z \rangle$ if $s \in V(Z)$. The ownership of the other vertices does not matter.

If a play is finite and ends in a vertex $\langle s, \psi \rangle$ then V wins if

- $\psi = P$ and $s \in \rho(P)$, or
- $\psi = Z$ and Z is free in φ and $s \in V(Z)$, or
- $\psi = \Box\theta$ and there is no state $t \in S$ such that $\langle s, t \rangle \in E$.

and R wins if

- $\psi = P$ and $s \notin \rho(P)$, or
- $\psi = Z$ and Z is free in φ and $s \notin V(Z)$, or
- $\psi = \Diamond\theta$ and there is no state $t \in S$ such that $\langle s, t\rangle \in E$.

If V wins we state $s_0 \models_V^S \varphi$. If the play is infinite, then

- V wins if the unique fixed point variable X which occurs infinitely often and subsumes all other variables occuring infinitely often, is bound by a ν-operator.
- R wins if the unique fixed point variable X which occurs infinitely often and subsumes all other variables occuring infinitely often, is bound by a μ-operator.

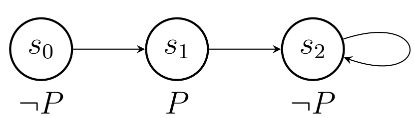
We can use either of these semantic notions as they are equivalent via:

Theorem 1. *$s \in \|\varphi\|_V^S$ iff V has a winning strategy for $\mathcal{G}_V^S(s, \varphi)$.*

We say that a formula φ is satisfiable iff there is a transition system S, a state $s \in S$ and a valuation V such that $s \in \|\varphi\|_V^S$.

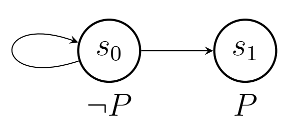
Before proceeding to show the basic properties of the μ-calculus we present a few examples.

Example 3. Let $\varphi = \mu X.P \vee \Diamond X$. Intuitively, this means "eventually P holds". Consider the following transition system S_1:



Here, $s_0 \models \varphi$, $s_1 \models \varphi$, and $s_2 \not\models \varphi$.

Example 4. Let $\psi = \nu X.\Diamond P \wedge \Box X$. Intuitively, this means "$\Diamond P$ always holds". Consider the following transition system S_2:



Here, $s_0 \not\models \psi$ and $s_1 \not\models \psi$.

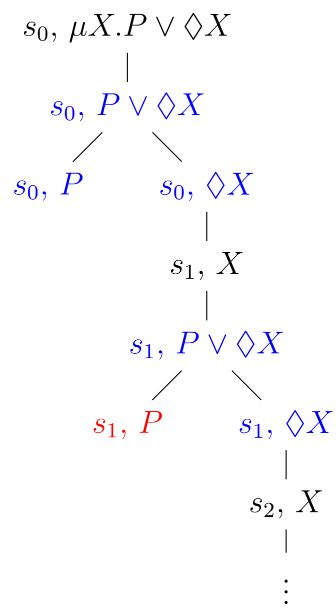


Figure 2.1: The game for $s_0 \models^{S_1} \varphi$. Vertices owned by V are blue and vertices owned by R are red.

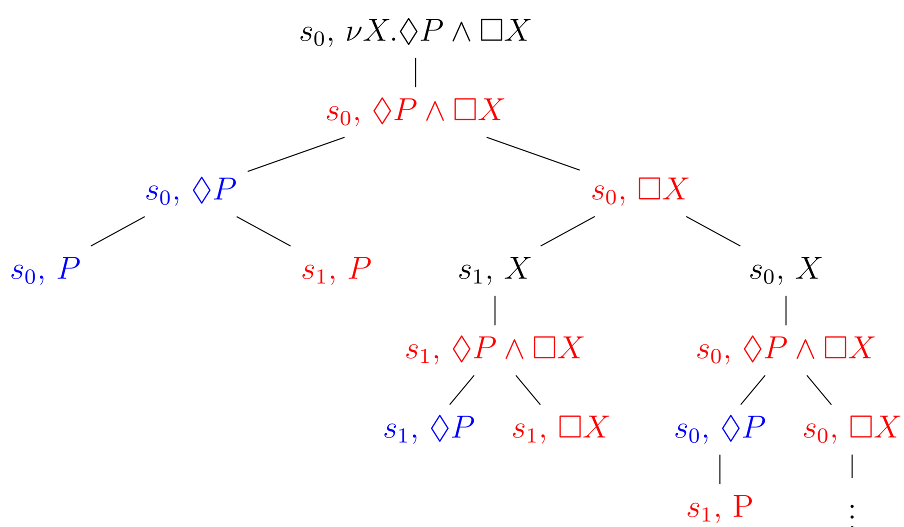


Figure 2.2: The game for $s_0 \not\models^{S_2} \psi$. Vertices owned by V are blue and vertices owned by R are red.

We sketch the games for $s_0 \models^{S_1} \varphi$ and $s_0 \models^{S_2} \psi$ on Figures 2.1 and 2.2, respectively.

By alternating μ and ν operators we can define more complex formulas. We can show that there are some formulas that actually need the alternation of operators to be defined. Let us now explain how to make this idea precise.

We start by defining how we count the operator alternations:

Definition 4 ((Emerson-Lei) Alternation Hierarchy)**.** Let $\alpha < \omega_1^{ck}$, then:

- Σ_0^μ, Π_0^μ: the class of formulas with no fixpoint operators
- $\Sigma_{\alpha+1}^\mu$: the class of formulas containing $\Sigma_\alpha^\mu \cup \Pi_\alpha^\mu$ and closed under the operations $\vee, \wedge, \Box, \Diamond, \mu X$ and the substitution: For a $\varphi(X) \in \Sigma_{\alpha+1}^\mu$ and a closed $\psi \in \Sigma_{\alpha+1}^\mu$, $\varphi(X \backslash \psi) \in \Sigma_{\alpha+1}^\mu$.
- $\Pi_{\alpha+1}^\mu$: the class of formulas containing $\Sigma_\alpha^\mu \cup \Pi_\alpha^\mu$ and closed under the operations $\vee, \wedge, \Box, \Diamond, \nu X$ and the substitution: For a $\varphi(X) \in \Pi_{\alpha+1}^\mu$ and a closed $\psi \in \Pi_{\alpha+1}^\mu$, $\varphi(X \backslash \psi) \in \Pi_{\alpha+1}^\mu$.
- Σ_λ^μ : the class of formulas containing $\bigcup_{\alpha<\lambda} \Sigma_\alpha^\mu$ and closed under $\bigvee_{i<\omega} \varphi_i$ where φ_i are recursively many formulas.
- Π_λ^μ : the class of formulas containing $\bigcup_{\alpha<\lambda} \Pi_\alpha^\mu$ and closed under $\bigwedge_{i<\omega} \varphi_i$ where φ_i are recursively many formulas.
- $\Delta_\alpha^\mu := \Sigma_\alpha^\mu \bigcap \Pi_\alpha^\mu$

We can get other hierarchies by modifying the above definition: by ommitting the substitution rule we get the simple alternation hierarchy $\Sigma_\alpha^{S\mu}$; and by changing the substition rule to require just that no free variable of ψ becomes a bound variable in $\varphi(X \backslash \psi)$ we get the Niwińsky hierarchy $\Sigma_\alpha^{N\mu}$. If we do not specify which of these hierarchies we are refering to, it means that the result holds for all of them. It is immediate from our definitions that:

Proposition 2. *For all $\alpha < \omega_1^{ck}$, $\Sigma_\alpha^{S\mu} \subsetneq \Sigma_\alpha^\mu \subsetneq \Sigma_\alpha^{N\mu}$.*

Here, ω_1^{ck} denotes the least non-computable ordinal.

As the hierarchies defined above are related only to the syntax of the μ-calculus we also define a semantic alternation hierarchy. Again we denote it by Σ_α^μ as this will not cause any confusion.

Definition 5 (Semantic Alternation Hierarchy)**.** For all $\alpha < \omega_1^{ck}$, we define

$$\Sigma^\mu_\alpha = \{\|\varphi\|^S | \ \varphi \in \Sigma^\mu_\alpha \text{ is a sentence and } S \text{ is a transition system}\}.$$

We have by [3] that

Theorem 3. *The semantic alternation hierarchy of the μ-calculus is strict.*

If we restrict the considered transition systems, the above theorem may hold or not. We say that a transition system (S, E, ρ) is finite iff S is a finite set. In the case of finite transition systems, Theorem 3 holds because:

Proposition 4. *If φ is a formula of the μ-calculus and is satisfiable, then φ is satisfiable by a finite model.*

We also consider recursively presentable transition systems (for short, *r.p.t.s*), i.e., transitions systems of the form (S, E, ρ) which each of S and E can be recursively coded as sets of integers and ρ is recursive. For simplicity, we consider S to be a recursive set of natural numbers. In this case, Theorem 3 also holds.

Figure 2.3 summarizes the results of this section:

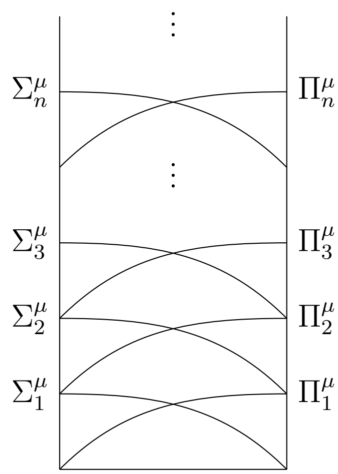


Figure 2.3: The altenation hierarchy of the μ-calculus.

Note that we have choosen to use only one modality in our μ-calculus. When applying the μ-calculus it is often useful to consider more modalities. All the results in this section also hold in this case.

2.3 μ-arithmetic

In this section we define an arithmetical variation of the μ-calculus. It is obtained by adding the fixed point operators μ and ν to the first-order arithmetic. In this context, $\mu x X.\varphi$ is the least fixed point of the operator $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $\Gamma(X) = \{x \in \omega | \varphi(x, X)\}$, and $\nu x X.\varphi$ is the greatest fixed point of this same operator. Note that μ and ν are dual by $\neg \mu x X.\varphi(X) = \nu x X.\neg\varphi(\neg X)$.

Example 5. The following formula defines the even numbers in the μ-calculus:

$$\mu x X.(x = 0 \vee (x - 2) \in X)$$

Calculating the least fixed point of $\Gamma_{x=0\vee(x-2)\in X}$ we have:

$$\emptyset \mapsto \{0\} \mapsto \{0, 2\} \mapsto \{0, 2, 4\} \mapsto \cdots \mapsto \{0, 2, 4, 6, 8, \cdots\}$$

By negation, we get that the odd numbers are defined by:

$$\nu x X.(x \neq 0 \wedge (x - 2) \in X)$$

Again, alculating the greatest fixed point of $\Gamma_{x\neq0\wedge(x-2)\in X}$ we have:

$$\omega \mapsto \omega \setminus \{0\} \mapsto \omega \setminus \{0, 2\} \mapsto \omega \setminus \{0, 2, 4\} \mapsto \cdots \mapsto \{1, 3, 5, 7, 9 \ldots\}$$

Before defining the μ-arithmetic we consider a concept we could evade with our definition of μ-calculus. Via the De Morgan dualities, we can push the negation symbols $\neg$ inwards so that it applies only to atomic formulas. We say φ is X-positive if all of the occurrences of X after applying this procedure are of the form $\tau \in X$, i.e., we only check positive ocurrences of terms in X. This is necessary to guarantee that the operator Γ is monotone, and so has least and greatest fixed points.

We define the μ-arithmetic and its alternation hierarchy simultaneously:

Definition 6. For each $\alpha < \omega_1^{ck}$:

- Σ_0^μ is the set of all set variables and formulas without fixpoint operators.
- $\Sigma_{\alpha+1}^\mu$ is generated from $\Sigma_\alpha^\mu \cup \Pi_\alpha^\mu$ by closing it under $\vee$, $\wedge$, $\in$ and $\mu x X.\varphi$ for X-positive $\varphi \in \Sigma_{\alpha+1}^\mu$. Here $\mu x X.\varphi$ is called a $\Sigma_{\alpha+1}^\mu$ term.
- $\Pi_{\alpha+1}^\mu$ contains all the negations of fomulas and set terms in $\Sigma_{\alpha+1}^\mu$.

- If λ is a limit ordinal, then Σ^μ_λ is generated from $\bigcup_{\alpha<\lambda}\Sigma^\mu_\alpha$ and closed under $\bigvee_{i<\omega}$.
- Π^μ_λ contains all the negations of Σ^μ_λ formulas and terms.

Note that when using the μ-operator we bind a number variable x and a set variable X. As in the μ-calculus, we say that the occurences of x or X in $\mu xX.\varphi$ are bound, and if x or X is not bound they are free. If a formula has no free *set* variable we say it is closed. Note that, in the case of μ-arithmetic, $\mu xX.\varphi$ denotes not a formula but a set term, and when constructing formulas with set terms they are syntactically the same as set variables. As $\mathbb{N}$ is the only intended model for μ-arithmetic, we do not need to define a semantic alternation hierarchy. We say a μ-formula is Σ^μ_α-definable iff it is equivalent to a Σ^μ_α-formula. And we say a μ-term is Σ^μ_α-definable iff it is equal to some Σ^μ_α μ-term. If $\varphi \in \Sigma^\mu_0$, we say it is arithmetical. Also note that we could possibly define Emerson-Lei and Niwińsky versions for the μ-arithmetic alternation hierarchy, but in this case all three hierarchies are equal, for details see section 4 of [3]. We can also check the validity of formulas of the μ-arithmetic by games, although we omit the definition of these games here they follow the same idea as games for the μ-calculus. We call these games model checking games. We also call the formulas of μ-arithmetic by μ-formulas, the intended meaning will be clear in context. If necessary we say that a formula of μ-calculus is a modal μ-formula and that a formula of μ-arithmetic is an arithmetic μ-formula.

We justify the increasing overload of meanings for the symbol Σ^μ_α by the following transfer theorems:

Theorem 5. *Let $\varphi(z)$ be a Σ^μ_n formula of μ-arithmetic. There is an r.p.t.s. T, a valuation V and a Σ^μ_n modal μ-formula $\overline{\varphi}$ such that $\varphi(s)$ iff $s \in \|\overline{\varphi}\|^T_V$.*

Theorem 6. *For each modal μ-calculus formula $\varphi \in \Sigma^\mu_n$ and for each recursively presentable transition system T, $\|\varphi\|^T$ is Σ^μ_n-definable set of integers.*

In Theorem 6, recall that we consider the set of states of an *r.p.t.s* to be a recursive set of integers, so $\|\varphi\|^T$ is a set of integers.

We have by [3] that

Theorem 7 (Bradfield)**.** *The alternation hierarchy of the μ-arithmetic is strict.*

This result is obtained by codifying the satisfiability of the μ-formulas inside μ-arithmetic. This proof uses the same idea as the proof for the strictness of the Arithmetic Hierarchy for PA, for more information see [2, 12]. Even though Bradfield's result says only about the finite levels of the alternation hierarchy, it can be extended to the full μ-arithmetic alternation hierarchy using the method described here. It can also be obtained as a corollary of Theorem 15.

An essential point in Bradfield's proof is the existence of a normal form for μ-formulas in the finite levels of the alternation hierarchy, a result first obtained by Lubarsky in [16]. This result will be useful for us in Chapter 3.

Theorem 8 (Lubarsky)**.** *Any Σ_n^μ-formula can be put in the form*

$$\tau_n \in \mu x_n X_n.\tau_{n-1} \in \nu x_{n-1} X_{n-1}.\tau_{n-2} \in \mu x_{n-2} X_{n-2}.\ldots.\tau_1 \in \eta x_1 X_1.\varphi$$

where φ is a μ-formula without set quantifiers and μ and ν operators and each τ_i is a (number) term. That is, for any Σ_n^μ-formula there is an equivalent formula that is generated by taking an arithmetical formula and alternating μ and ν operators.

2.4 Gale-Steward Games

In this section we introduce Gale-Steward games in the Baire space. We also present some basic results about determinacy. In this section we use some basic definitions of descriptive set theory. As general references for descriptive set theory and Gale-Steward games, see [9, 19].

The Baire space is the space ω^ω of all infinite sequences of natural numbers with the product topology. Let s be a finite sequence of natural numbers of length n. The open interval $[s]$ is the set $\{x \in \omega^\omega | x \upharpoonright n = s\}$. The open intervals form an open basis of the topology of ω^ω.

Definition 7. A game on the Baire space is defined as follows: Two players (I and II) alternate playing elements from ω to form a sequence in ω^ω. This sequence is called a run. I wins a run x of the game iff $x \in A$ where A is a fixed subset of ω^ω. Otherwise II wins. A is called the payoff of the game. We denote this game by G_A. A strategy for I is a function from the finite sequences of ω with even length into ω. I plays a run x with strategy σ iff for all $n \in \omega$ $x(n) = \sigma(x \upharpoonright n)$. If σ is a strategy for I and $z \in X^\omega$, the run where I plays according to σ and II plays z is denoted $\sigma(z)$. A strategy for I is winning iff for all $z \in X^\omega$ $\sigma(z) \in A$. We define the notions of strategy for II similarly. A game is determined iff either I or II has a winning strategy.

In case the payoff set A is open, $\mathbf{\Sigma^0_2}$, Borel, etc., we say the game G_A is open, $\mathbf{\Sigma^0_2}$, Borel, etc.

Having defined games on ω^ω, we can state:

Definition 8 (AD)**.** The Axiom of Determinacy states that for every $A \subset \omega^\omega$ the game G_A is determined.

Theorem 9. *$ZF + AD + AC \vdash \perp$,*

By the above theorem, we can not assume AD unless we abstain from supposing Choice. But assuming choice we still have:

Theorem 10 (Martin)**.** *$ZFC \vdash$ all Borel games are determined.*

All the games used in the semantics for the μ-calculus and μ-arithmetic have Borel payoff, we work inside ZFC.

As a historical sidepoint we state two subcases of Borel Determinacy. We will reference the proofs of these theorems in Chapters 3 and 4.

Theorem 11 (Gale-Steward)**.** *$ZFC \vdash$ all open games are determined.*

Theorem 12 (Wolfe)**.** *$ZFC \vdash$ all $\mathbf{\Sigma^0_2}$ games are determined.*

We make one more definition before going to the next topic:

Definition 9. We define the quantifier $\Game$ by

$\Game\alpha.P(\alpha, \vec{x}) = \{\vec{x} |\ I$ wins the Gale-Steward game with payoff $P(\alpha, \vec{x})\} \subseteq \omega^k$.

where $P \subseteq \omega^\omega \times \omega^k$ for some $k \in \omega$.

If Γ is a pointclass, define $\Game\Gamma = \{S | S = \Game\alpha.P(\alpha, \vec{x})$ where $P \in \Gamma\}$. For context, the following holds:

Theorem 13 (Kechris,Moschovakis)**.** $\Game\Sigma^0_1 = \Pi^1_1$.

Theorem 14 (Solovay)**.** *$\Game\Sigma^0_2 = \Sigma^1_1$-IND. Here, Σ^1_1-IND is the class of sets given via an inductive definition over a Σ^1_1 predicate.*

2.5 The Difference Hierarchy

In this section we the difference hierarchy of Σ_2^0 and its connection to the μ-arithmetic. This result is from [5–7].

Recall that the difference hierarchy for Σ_2^0 is defined as follows:

Definition 10. For each $\alpha < \omega_1^{ck}$,

$$S \in \Sigma_\alpha^\delta \iff S = \bigcup_{\beta \in Opp(\alpha)} (A_\beta - \cup_{\zeta<\beta} A_\zeta)$$

where $(A_\beta)_{\beta<\alpha}$ is an effective enumeration of a sequence of sets in Σ_2^0 and $Opp(\alpha)$ is the set of ordinals less that α whose parity is opposite to the parity of α. (We consider the limit ordinals to be even.)

Note that if we substitute Σ_2^0 by other pointclasses we can get other alternation hierarchies.

For the finite levels of the difference hierarchy we can consider the following alternative definition:

Definition 11. Let $n \in \omega$, then:

- $\Sigma_0^\delta = \Sigma_1^0$,
- $\Pi_n^\delta = \neg\Sigma_n^\delta$, and
- $\Sigma_{n+1}^\delta = \Sigma_2^0 \wedge \Pi_n^\delta$.

Now we can state the main theorem from [5–7] which connects the difference hierarchy to the alternation hierarchy of the μ-arithmetic:

Theorem 15 (Bradfield, Duparc, Quickert)**.** *For all* $\alpha < \omega_1^{ck}$, $\Game\Sigma_\alpha^\delta = \Sigma_{\alpha+1}^\mu$.

From [17], we have:

Theorem 16 (MedSalem, Tanaka)**.** $\bigcup_{\alpha<\omega_1^{ck}} \Sigma_\alpha^\delta = \Delta_3^0$

By combining the two theorems above and noting that $\cup_{i<\omega}$ commutes with $\Game$, we have:

Corollary 17. $\bigcup_{\alpha<\omega_1^{ck}} \Sigma_\alpha^\mu = \Game\Delta_3^0$.

2.6 Reverse Mathematics

In this section, we shortly explain Reverse Mathematics and present the basic results relating to determinacy. Reverse Mathematics aims to answer the question of what axioms are necessary for theorems of "ordinary" mathematics. For a throughout introduction see [21].

We work here with the second order arithmetic Z_2, which is a theory in a two-sort first-order logic describing the natural numbers and sets of natural numbers. We call its language $\mathcal{L}_2$. It is a quite strong theory, and we are able to show many theorems of ordinary mathematics inside it. Indeed, it is *too* strong for our objectives, so we consider some subsystems of Z_2. The five main subsystems of Z_2 are:

- RCA_0 states the basic axioms for the ordering of the natural numbers, natural addition and multiplication, plus induction for Σ^0_1 formulas and comprehension for Δ^0_1 formulas.
- WKL_0 is RCA_0 plus the statement that every infinite binary tree has an infinite path.
- ACA_0 is RCA_0 plus comprehension for all arithmetic formulas.
- ATR_0 is ACA_0 plus axioms for "roughly" transfinite induction on arithmetic formulas.
- Π^1_1-CA_0 is ACA_0 plus comprehension for Π^1_1 formulas.

The systems above are in order of strictly increasing strength. We work over some base system (usually RCA_0 or ACA_0) and show that some other system and some theorem are equivalent over the base system. Many of the theorems of ordinary mathematics are equivalent to some of the five systems above, and also systems stronger than RCA_0 but weaker that Π^1_1-CA_0. There are not many examples of theorem of ordinary mathematics above Π^1_1-CA_0.

We present a few results related to determinacy in Reverse Mathematics in the following theorem. For more details, see [23].

Theorem 18. *The following statements hold:*

- *(Steel)* $\mathrm{ACA}_0 \vdash \mathrm{ATR}_0 \leftrightarrow \Delta^0_1$ *determinacy* $\leftrightarrow \Sigma^0_1$ *determinacy.*
- *(Tanaka)* $\mathrm{ATR}_0 \vdash \Pi^1_1$-$\mathrm{CA}_0 \leftrightarrow \Pi^0_1 \wedge \Sigma^0_1$ *determinacy.*

- *(Montalbán-Shore)* $\Pi^1_{n+2} CA_0 \vdash n\text{-}\Sigma^0_3$ *determinacy.*
- *(Montalbán-Shore)* $\Delta^1_{n+2} CA_0 \nvdash n\text{-}\Sigma^0_3$ *determinacy.*
- *(Montalbán-Shore)* $Z_2 \nvdash (\Sigma^0_3)_{<\omega}$ *determinacy.*

Here $n\text{-}\Sigma^0_3$ *denotes the* n*-th level of the difference hierarchy for* Σ^0_3*, and* $(\Sigma^0_2)_{<\omega}$ *denotes the union* $\bigcup_{n\in\omega} n\text{-}\Sigma^0_3$*.*

Chapter 3

The weak μ-arithmetic

3.1 The weak μ-arithmetic and the weak μ-calculus

In this section we present a weak version of the μ-arithmetic and the μ-calculus and some of its basic properties. This weak version is obtained by restricting the way we can alternate the μ and ν operators. This definition comes from Li's thesis [15].

Definition 12. We define the weak alternation hierarchy for the μ-arithmetic as follows:

- $\Sigma_0^{W\mu}$ is the set of all the first order formulas and all set variables.
- $\Sigma_{\alpha+1}^{W\mu}$ is generated from $\Sigma_\alpha^{W\mu} \cup \Pi_\alpha^{W\mu}$ by closing it under $\vee$, $\wedge$ and the following substitution rules: (a) If $\varphi(X)$ is Σ_1^μ and if ψ is a $\Sigma_{\alpha+1}^{W\mu}$ term without free set variables, then $\varphi(X\backslash\psi)$ is also $\Sigma_{\alpha+1}^{W\mu}$; (b) if φ is a Σ_1^μ, φ' is a subformula of φ and ψ is a $\Sigma_{\alpha+1}^{W\mu}$ term without free set variables,then $\varphi(\varphi'\backslash\psi)$ is also $\Sigma_{\alpha+1}^{W\mu}$. In these substitution rules, φ can be either a formula or a term.
- If λ is a limit ordinal, then $\Sigma_\lambda^{W\mu}$ is generated from $\bigcup_{\alpha<\lambda} \Sigma_\alpha^{W\mu}$ and closed under $\bigvee_{i<\omega}$.
- $\Pi_\alpha^{W\mu}$ contains all the negations of fomulas and set terms in $\Sigma_\alpha^{W\mu}$.
- $\Pi_\lambda^{W\mu}$ contains all the negations of $\Sigma_\lambda^{W\mu}$ formulas and terms.

Observe that we abuse the notation of substitution in this definition. This is necessary in the transfinite levels of the weak hierarchy, as there are no weak μ-term strictly in the limit levels.

Definition 13. We define the weak alternation hierarchy for the μ-calculus as follows:

- $\Sigma_0^{\mathrm{W}\mu}, \Pi_0^{\mathrm{W}\mu}$: the class of formulas with no fixpoint operators
- $\Sigma_{\alpha+1}^{\mathrm{W}\mu}$: the class of formulas containing $\Sigma_\alpha^{\mathrm{W}\mu} \cup \Pi_\alpha^{\mathrm{W}\mu}$ and closed under the operations $\vee, \wedge, \Box, \Diamond$ and the substitution: For a $\varphi(X) \in \Sigma_1^\mu$ and a closed $\psi \in \Sigma_{\alpha+1}^{\mathrm{W}\mu}$, $\varphi(X \backslash \psi) \in \Sigma_{\alpha+1}^{W\mu}$.
- Dually for $\Pi_{\alpha+1}^\mu$
- If λ is a limit ordinal, then $\Sigma_\lambda^{W\mu}$ is generated from $\bigcup_{\alpha<\lambda} \Sigma_\alpha^{W\mu}$ and closed under $\bigvee_{i<\omega}$.
- Dually for Π_λ^μ
- $\Delta_\alpha^{W\mu} := \Sigma_\alpha^{\mathrm{W}\mu} \bigcap \Pi_\alpha^{\mathrm{W}\mu}$

These alternations are weak in the sense that we are restricting the interaction between the μ and ν operators. Indeed, the weak definable sets are relatively low on the full alternation hierarchy. For the first ω levels it is easy to check that:

Proposition 19. *For all $n \in \omega$, $\Sigma_n^{\mathrm{W}\mu} \subsetneq \Delta_2^\mu$.*

For the μ-calculus this only holds for the Emerson-Lei and Niwińsky alternation hierarchies, and the proof is simple. For the μ-arithmetic, we note that the simple and Niwińsky hierarchies coincide, as we said in Section 2.3. This does hold for the transfinite levels, but this is not so easy because of the recursive disjunction and conjunction and we need to look at the semantics side of the problem.

In the previous chapter we presented Theorems 5 and 6. There are transfer results between the μ-arithmetic and the μ-calculus. These transfer results also hold in the weak context:

Theorem 20. *Let $\varphi(z)$ be a $\Sigma_\alpha^{W\mu}$ formula of μ-arithmetic. There is a r.p.t.s. T, a valuation V and a $\Sigma_\alpha^{W\mu}$ modal μ-formula $\overline{\varphi}$ such that $\varphi(s)$ iff $s \in \|\overline{\varphi}\|_V^T$.*

Theorem 21. *For each modal μ-calculus formula $\varphi \in \Sigma_\alpha^{W\mu}$ and for each recursively presentable transition system T, $\|\varphi\|^T \subseteq \omega$ is $\Sigma_\alpha^{W\mu}$-definable set of integers.*

Proof. Observe that, if we start with weak formulas, the new formulas defined in Bradfield's proof are also weak. Therefore these theorems follow from the proof of the full case. $\square$

In this chapter, we consider only the semantic alternation hierarchy for the weak μ-calculus restricted to recursively presented transition systems:

Definition 14. For all $\alpha < \omega_1^{ck}$, define

$$\Sigma_\alpha^{W\mu} = \{\|\varphi\|^T \subseteq \omega | \varphi \in \Sigma_\alpha^{W\mu} \text{ and } T \text{ is an r.p.t.s}\}$$

In the following result, we denote by $\Sigma_{\alpha,A}^{W\mu}$ the sets of integers definable by $\Sigma_\alpha^{W\mu}$-formulas of μ-arithmetic and by $\Sigma_{\alpha,C}^{W\mu}$ the levels of the semantic alternation hierarchy for the μ-calculus. By theorems 20 and 21, we have:

Corollary 22. *For all $\alpha < \omega_1^{ck}$, $\Sigma_{\alpha,A}^{W\mu} = \Sigma_{\alpha,C}^{W\mu}$.*

That is, it does not matter if we work on the μ-arithmetic or the μ-calculus if we are interested in the alternation hierarchy.

By adapting Bradfield's proof in [2], we can get the strictness of the finite levels of the weak alternation hierarchies:

Theorem 23. *The weak alternation hierarchies for the μ-calculus and the μ-arithmetic are strict below ω.*

Proof. We can adapt the proof for the full μ-arithmetic that is in [2]. This is a generalization of proof of the strictness of the arithmetic hierarchy in PA using satisfaction formulas. We omit this proof as it is rather complicated but not conceptually hard. For the μ-calculus we use the result for μ-arithmetic and Theorems 20 and 21. $\square$

We conjecture that this proof can be extended to the transfinite levels.

The following normal form for the weak μ-arithmetic is going to be useful in the next section:

Theorem 24. *Every weak μ-formula and μ-term can be put in a normal form:*

- *If* $\varphi \in \Sigma_1^{W\mu}$ *is a term, then* $\varphi \equiv \mu X_1.\psi$ *where* ψ *is an arithmetical formula.*

- *If* $\varphi \in \Sigma_{\alpha+1}^{W\mu}$ *is a term, then* $\varphi \equiv \mu X_{\alpha+1}.\psi(\psi_1,\ldots,\psi_n)$ *where* ψ *is an arithmetical formula and* $\psi_1,\ldots,\psi_n$ *are* $\Pi_\alpha^{W\mu}$ *terms in normal form.*

- *If* α *is not a limit ordinal and* $\varphi \in \Sigma_\alpha^{W\mu}$ *is a formula, then* $\varphi \equiv \tau_\alpha \in \varphi'$ *where* $\varphi \in \Sigma_\alpha^{W\mu}$ *is a term in normal form.*

- *If* λ *is a limit ordinal and* $\varphi \in \Sigma_\lambda^{W\mu}$ *is a formula, then* $\varphi \equiv \bigvee_{n\in\omega}\psi_n$ *where each* ψ_n *is a formula in* $\bigcup_{\alpha<\lambda}\Sigma_\alpha^{W\mu}$*. We furthemore suppose that* X_β *is a* μ*-variable iff* α *and* β *have the same parity.*

Note that if λ *is a limit ordinal, then there are no* μ*-terms in* $\Sigma_\lambda^{W\mu}\setminus\bigcup_{\alpha<\lambda}\Sigma_\alpha^{W\mu}$*.*

Proof. Use Lubarsky's Normal Form Theorem for μ-formulas and observe that using the available arithmetical machinery we can do all the substitutions at the same time. We can resolve the parity condition on the limit case by padding the terms with μ and ν operators. □

3.2 μ-arithmetic and the difference hierarchy

In this section, we prove the weak version of Theorem 15. Here $\Sigma_\alpha^{\delta,1}$ denotes α-th level of the difference hierarchy for Σ_1^0 sets of finite sequences of integers.

Theorem 25. $\Sigma_{\alpha+1}^{W\mu} = \Game\Sigma_\alpha^{\delta,1}$*, for all* $\alpha < \omega_1^{ck}$*.*

Proof. For clearness, we divide this proof into two claims. Both of them are proofs by induction.

Claim 1. $\Game\Sigma_\alpha^{\delta^1} \subseteq \Sigma_{\alpha+1}^{W\mu}$*, for all* $\alpha < \omega_1^{ck}$*.*

Proof. We prove this based on Gale and Steward's proof of Open Determinacy. This proof was inspired by [5–7].

Before formalizing the proof we sketch its idea. The proof of Open Determinacy proceeds by defining the winning position of the game

$$\exists n.Q(\alpha[n],\vec{x})$$

where Q is a recursive set and $\vec{x}$ are natural number parameters. We omit the parameters most of the time. We start defining the winning positions with the "easy" positions:

$$s \in W_0 \text{ iff } Q(s)$$

If we have defined W_α, we define $W_{\alpha+1}$ by adding to W_α all the positions where I can immediately force the play into W_α, i.e.,

$$\begin{aligned} s \in W_{\alpha+1} \text{ iff } s \in W_\alpha & \\ & \vee \; lh(s) \text{ is odd and } \forall n. s \frown n \in W_\alpha \\ & \vee \; lh(s) \text{ is even and } \exists m \forall n. s \frown m \frown n \in W_\alpha \end{aligned}$$

For a limit ordinal λ, we define $W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$.

We can extend this proof to the difference hierarchy for Σ^0_1 by adding a $\Pi^{\delta,1}_\alpha$ parameter to the open game. This results in the game

$$\exists n. Q(\alpha[n]) \wedge R(\alpha).$$

If we denote the winning region of R by W_R we define the winning region of the new game by:

$$s \in W_0 \text{ iff } Q(s) \wedge s \in W_R$$

$$\begin{aligned} s \in W_{\alpha+1} \text{ iff } s \in W_\alpha & \\ & \vee \; lh(s) \text{ is odd and } \forall n. s \frown n \in W_\alpha \cap W_R \\ & \vee \; lh(s) \text{ is even and } \exists m \forall n. s \frown m \frown n \in W_\alpha \cap W_R \end{aligned}$$

We can then show that

- There is a winning strategy for I iff $\langle\rangle \in W$.
- There is a winning strategy for II iff $\langle\rangle \notin W$.

We now formalize the argument above in the μ-arithmetic by defining the winning region inside it.

- Let $\alpha = 0$. We show that $\Game\Sigma_0^{\delta,1} \subseteq \Sigma_1^{W\mu}$. Consider the open game $P(\alpha, \vec{x}) = \exists n.Q(\alpha(n))$. We define its winning positions by:

$$\begin{aligned} W_P = \mu y Y.(&Q(y) \\ &\vee (lh(y) \text{ is odd and } \forall n.s \frown n \in Y) \\ &\vee (lh(y) \text{ is even and } \exists m \forall n.s \frown m \frown n \in Y)) \end{aligned}$$

  We have then that $\vec{x} \in \Game\alpha.P(\alpha, \vec{x})$ iff $\langle\rangle \in W(\vec{x})$.

- We show that $\Game\Sigma_{\alpha+1}^{\delta,1} \subseteq \Sigma_{\alpha+2}^{W\mu}$. Consider the game $P(\alpha, \vec{x}) = \exists n.Q(\alpha(n)) \wedge R(\alpha)$, where Q is recursive and R is $\Pi_\alpha^{\delta,1}$. We define its winning positions by:

$$\begin{aligned} W_P = \mu y Y.((&Q(y) \wedge W_R(y)) \\ &\vee (lh(y) \text{ is odd and } \forall n.y \frown n \in Y) \\ &\vee (lh(y) \text{ is even and } \exists m \forall n.y \frown m \frown n \in Y)) \end{aligned}$$

  We have then that $\vec{x} \in \Game\alpha.P(\alpha, \vec{x})$ iff $\langle\rangle \in W(\vec{x})$ again.

- Let λ be a limit ordinal and suppose $\Game\Sigma_\alpha^{\delta,1} \subseteq \Sigma_{\alpha+1}^{W\mu}$ holds for all $\alpha < \lambda$. Let $P(\alpha, \vec{x}) = \bigvee_{i<\omega} \varphi_n$ be a $\Sigma_\lambda^{\delta,1}$ game, with each φ_i in $\bigcup_{\alpha<\lambda} \Sigma_\lambda^{\delta,1}$. Then

$$\begin{aligned} W = \mu y Y.(&\bigvee_{n<\omega} W_{\varphi_n}(y)) \\ &lh(y) \text{ is odd and } \forall n.s \frown n \in Y \\ &lh(y) \text{ is even and } \exists n \forall n.s \frown m \frown n \in Y) \end{aligned}$$

  Yet another time, $\vec{x} \in \Game\alpha.P(\alpha, \vec{x})$ iff $\langle\rangle \in W(\vec{x})$.

We can then conclude that $\Game\Sigma_\alpha^{\delta,1} \subseteq \Sigma_{\alpha+1}^{W\mu}$ for all $\alpha < \omega_1^{ck}$. □

Claim 2. *$\Sigma_{\alpha+1}^{W\mu} \subseteq \Game\Sigma_\alpha^{\delta,1}$, for all $\alpha < \omega_1^{ck}$.*

Proof. This proof is based on the proof for Claim 2 of [7]. Let $\varphi(\vec{x})$ denote a $\Sigma_{\alpha+1}^{W\mu}$ term in normal form (check Theorem 24). The key fact is that the model checking game for a $\Sigma_{\alpha+1}^{W\mu}$-formula is essentially the code of a $\Sigma_\alpha^{\delta,1}$ set. In case of $\alpha = 0$, if I wins the model checking game G for $n \in \varphi(\vec{x})$ he wins in finite time, so G is an open game. For the other cases we use a more sofisticated argument which holds for the successor and limit cases.

Suppose there is some $n \in \varphi$. Consider the model checking game for $n \in \varphi$. We can consider this game to be a subtree of ω^* by coding. Furthermore, we can suppose that this tree has no finite maximal branches. To simplify the tree, we suppose each node of the tree marks a loopback, i.e., that each node is some $n' \in X_\beta$. We also ommit the n's and keep track only of the βs, so each vertice is just an ordinal number β. We denote the tree obtained by this process by T.

For each successor $\beta \in \alpha$, we define

$$C_\beta = \{x \in [T] | \exists n.x(n) = \beta\}$$

Each C_β is in Σ_1^0. Let $I = \{\beta < \alpha | parity(\beta) \neq parity(\alpha)\}$. Define

$$C = \bigcup_{\beta \in I} C_\beta \setminus (\bigcup_{\zeta < \beta} C_\zeta)$$

C is a $\Sigma_\alpha^{\delta,1}$ set. We show C is the payoff set for I.

Let $x \in C$ and fix the β such that $x \in C_\beta \setminus (\bigcup_{\zeta<\beta} C_\zeta)$. We have that the β with parity not equal to the parity of α are all ν-variables, by the definition of the normal form. Therefore x is a run that eventually loops the ν-variable X_β. That is, x is a winning run for I.

If x is a run that is winning for I, then it eventually inspects only some ν-variable X_β, so $x \in C_\beta \setminus (\bigcup_{\zeta<\beta} C_\zeta)$ and consequently $x \in C$. □

This concludes the proof of Theorem 25. □

The following is a result by Tanaka[22]:

Theorem 26 (Tanaka)**.** $\bigcup_{\alpha<\omega_1^{ck}} \Sigma_\alpha^{\delta,1} = \Delta_2^0$.

As a consequence we have:

Corollary 27. $\bigcup_{\alpha<\omega_1^{ck}} \Sigma_\alpha^{W\mu} = \Game\Delta_2^0$.

By Theorem 15, we have that $\Game\Delta_2^0 = \Delta_2^\mu$, so the following also holds:

Corollary 28. $\bigcup_{\alpha<\omega_1^{ck}} \Sigma_\alpha^{W\mu} = \Delta_2^\mu$.

Figure 3.1 summarizes our results.

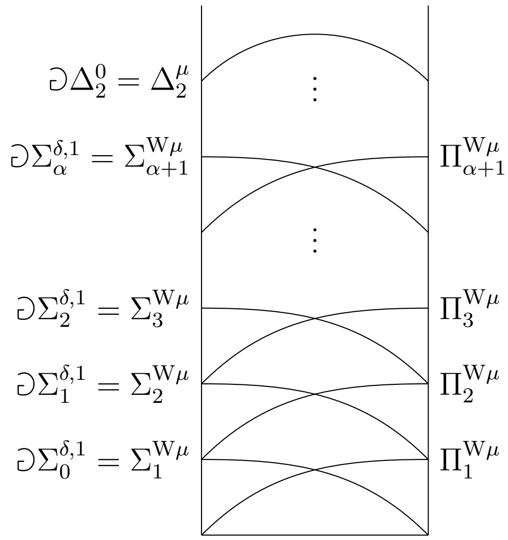


Figure 3.1: The weak altenation hierarchy of the μ-arithmetic.

Open Question 1. *Observe that the weak altenation hierarchy is a refinement of the altenation hierarchy up to Δ^μ_2. Can we extend this idea to the upper levels? For example, can we extend the weak altenation hierarchy to a hierarchy $\Sigma^{W\mu}_\alpha(\Sigma^\mu_2)$ such that*

$$\bigcup_{\alpha<\omega_1^{ck}} \Sigma^{W\mu}_\alpha(\Sigma^\mu_2) = \Delta^\mu_3$$

If we can do this, can we extend this further to all the ω_1^{ck} levels of the alternation hierarchy?

Open Question 2. *If we can solve the problem above, we would be able to obtain a finer view of the Wadge Degrees involved in the alternation hierarchy of μ-arithmetic.*

Chapter 4

μ-arithmetic and Second Order Arithmetic

In this chapter we present the proof that the formalized version of the μ-arithmetic inside $\mathcal{L}_2$ is equivalent to $(\Sigma^0_2)_{<\omega}$determinacy over ACA_0 and present a refinement for this result. This is the main theorem of [18].

As the proof of Heinatsch and Möllerfeld's theorem is quite long and involved, we only explain it to the extent that we can obtain our result. Before doing so, we explain the outline of this chapter. In Section 4.1, we formalize the μ-arithmetic and present the theory ⅁ame for describing games in second order arithmetic. These two theories are in fact equivalent over $\mathcal{L}_2$-formulas. In Section 4.2, we present the proof of Heinatsch and Möllerfeld's theorem. We divide this proof into two parts. To prove $(\Sigma^0_2)_{<\omega}$determinacy from μ-arithmetic we adapt Wolfe's proof of $\mathbf{\Sigma^0_2}$ determinacy. To embed ⅁ame inside $(\Sigma^0_2)_{<\omega}$determinacy we have to define some quite complex games, here we present only the definitions relevant to the proof of our result. In Section 4.3 we finally prove give the refinement of Heinatsch and Möllerfeld's theorem.

In this section, we use ω to denote the "real" natural numbers and $\mathbb{N}$ to denote the natural numbers inside our axiom systems.

4.1 μ-arithmetic and Reverse Mathematics

In this section we formalize the μ-arithmetic inside Z_2. These results are from [18].

We define the language $\mathcal{L}_\mu$ of μ-arithmetic by adding the constructor μ to $\mathcal{L}_2$. Define the set of $\mathcal{L}_\mu$-formulas and $\mathcal{L}_\mu$-terms to be the smallest set which includes the $\mathcal{L}_2$-formulas and is closed under the usual rules for forming $\mathcal{L}_2$-formulas and the following rule: if $\varphi(x, X)$ is an X-positive formula of $\mathcal{L}_\mu$, we add a set term $\mu x X.\varphi(x, X)$, with the restriction that $\varphi(x, X)$ has no second order quantifiers.

The term $\mu x X.\varphi(x, X)$ denotes the least fixed point of the operator $\Gamma_\varphi(X) = \{x|\varphi(x, X)\}$. To formalize this idea we define for each X-positive formula $\varphi(x, X)$ the following formula:

$$LFP(\varphi, I) \iff \forall x.(x \in I \leftrightarrow \varphi(x, I)) \wedge \forall Y.(\forall x.(\varphi(x, Y) \to x \in Y) \to I \subset Y$$

LFP stands for least fixed point and $LFP(\varphi, I)$ means that I is a fixed point of $\Gamma_\varphi(X)$ and is the least such fixed point.

Definition 15. The μ-arithmetic is the system contains the axioms of ACA_0 (including comprehension for $\mathcal{L}_\mu$-formulas with no set quantifiers) and contains $LFP(\varphi(x, X), \mu x X.\varphi(x, X))$ for each X-positive formula $\varphi \in \mathcal{L}_\mu$ with no set quantifiers. Note that we do not consider μ a set quantifier, so we can take fixed points of formulas which include μ.

We also define

$$\begin{aligned} IGF(\varphi, S, \preceq, \prec) \iff & (S, \preceq, \prec) \text{ is a prewell-ordering and} \\ & \forall x, y(x \preceq y \leftrightarrow x \prec y \vee \varphi(x, \{z|z \prec y\})). \end{aligned}$$

Moreover, over ACA_0, if $\varphi(x, X)$ is an X-positive formula, S is a set and $\prec, \preceq$ are binary relations on S, then $IGF(\varphi, S, \preceq, \prec)$ implies $LFP(\varphi, S)$.

A generalized quantifier $\mathbf{Q}$ is a subset of $\mathcal{P}(\mathbb{N})$ such that

$$\begin{aligned} &\emptyset \notin \mathbf{Q} \\ &\mathbf{Q} \neq \emptyset \\ &X \subset Y \wedge X \in \mathbf{Q} \Rightarrow Y \in \mathbf{Q}. \end{aligned}$$

We abbreviate $\{x|\varphi(x)\} \in \mathbf{Q}$ by $\mathbf{Q}x.\varphi(x)$. We define the inverse quantifier $\overline{\mathbf{Q}}$ by $\overline{\mathbf{Q}} = \{\neg X | X \notin \mathbf{Q}\}$. We have that $\forall = \{\omega\}$, $\exists = \mathcal{P}(\omega) \setminus \{\emptyset\}$, $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

The next quantifier or open game quantifier $\mathbf{Q}^\vee$ is defined by

$$\mathbf{Q}^\vee x.\varphi(x) \iff (\overline{\mathbf{Q}}x_0)(\overline{\mathbf{Q}}x_1)(\overline{\mathbf{Q}}x_2)(\overline{\mathbf{Q}}x_3)\cdots \bigvee_{n\in\omega} \varphi(\langle x_0, \ldots, x_n\rangle)$$

In this thesis we only use the following generalized quantifiers:

$$\exists^0 = \exists; \forall^n = \overline{\exists^n}; \exists^{n+1} = (\exists^n)^\vee$$

These quantifiers are not definable in $\mathcal{L}_2$, but we define an adequate extention to $\mathcal{L}_2$ in which all the quantifiers $\exists^n$ and $\forall^n$ are definable. We define $\mathcal{L}_{\Game}$ by adding the quantifier symbols $\exists^n$ and $\forall^n$ for all $n \in \omega$. The $\mathcal{L}_{\Game}$-formulas are defined the same way $\mathcal{L}_2$-formulas are defined, with the additional rule that $\exists^n x.\varphi(x)$ and $\forall^n x.\varphi(x)$ are valid formulas if and only if φ has no second-order quantifiers (even if $\exists^n$ and $\forall^n$ occur in φ).

Definition 16. The theory $\Game$ame (with language $\mathcal{L}_{\Game}$) consists of the following axioms:

- the axioms of ACA_0, with comprehension for all $\mathcal{L}_{\Game}$-formulas without second-order quantifiers.
- $\exists^0 x.\varphi(x) \leftrightarrow \exists x.\varphi(x)$
- $\exists^{n+1} x.\varphi(x, \vec{y}, \vec{Y}) \leftrightarrow \forall X(\forall x.(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \rightarrow x \in X) \rightarrow \langle\rangle \in X)$
- $\forall^n x.\varphi(x) \leftrightarrow \neg\exists^n x.\neg\varphi(x)$

where φ varies over formulas without second-order quantifiers.

Above we have that φ^Q is an abbreviation for

$$Qx.s ^\frown \langle x\rangle \notin X \rightarrow \varphi(s, \vec{y}, \vec{Y})$$

As φ^Q is X-positive, $\forall X(\forall x.(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \rightarrow x \in X) \rightarrow \langle\rangle \in X)$ expresses that $\langle\rangle$ is in the least fixed point operator defined by $\varphi^{\exists^n}(x, X)$.

We have from [18] that

Theorem 29. *The μ-arithmetic and $\Game$ame prove the same $\mathcal{L}_2$ sentences.*

4.2 μ-arithmetic is equivalent to $(\Sigma^0_2)_{<\omega}$determinacy

In this section we show that μ-arithmetic is equivalent to $(\Sigma^0_2)_{<\omega}$determinacy. This is a result from [10]. We first show that the μ-arithmetic proves $(\Sigma^0_2)_{<\omega}$determinacy.

Theorem 30. *Let G be a $(\Sigma_2^0)_{<\omega}$game. Then, inside the μ-arithmetic, we can uniformly define the set of winning positions for player I and strategies S_I^G and S_{II}^G. Furthermore, if $\langle\rangle \in W^G$ then S_I^G is a winning strategy for I in G and if $\langle\rangle \notin W^G$ then S_{II}^G is a winning strategy for II in G.*

Proof. The proof of this theorem is based on Wolfe's proof of $\mathbf{\Sigma_2^0}$ determinacy. It has the same structure as the proof of Claim 1 of Theorem 25. We omit it here, it can be found on [10] with full details.

□

In the other direction, we embed the theory ⅁ame in $(\Sigma_2^0)_{<\omega}$determinacy. This is sufficient by theorem 29. We describe here only the beginning of the proof, as it is the only part we need to analyse for our result.

Here we define a $\mathbf{\Sigma_{n-1}^{\delta}}$ game for each $\exists^n$, using this games we then embed ⅁ame inside $(\Sigma_2^0)_{<\omega}$determinacy. For this we need some auxiliary games, but the complexity of these will not matter for our theorem. Most of the proof is the complicated induction in Theorem 32.

Definition 17. For each $n \in \omega$, we define a game for $\exists^n$.

We start with the game for $\exists^1 = \exists^{\vee}$. Intuitively, $\exists^1 x.\varphi(x)$ means

$$\forall x_0 \forall x_1 \forall x_2 \cdots \bigvee_{n \in \mathbb{N}} \varphi(\langle x_0, \ldots, x_n \rangle).$$

(Strictly it is of the form $\forall f \exists c \varphi(f \restriction n)$.) In this game II plays natural numbers until I plays a break. If the sequence up to the break satisfies φ, then I wins; otherwise, I loses. If I does not play a break, I also loses. This is an open game as if I wins then it wins in finite time.

We define the game for $\forall^1$ by exchanging the roles of I and II.

We now define the game for $\exists^2$. It it starts by I choosing whether to play a $\forall^1$ game or not. The players play this game until II plays its first break. Then I repeats this process. If I chooses to play infinitely many $\forall^1$ games, then I loses. If I stops after the n^{th} game, I wins iff $\varphi(\langle s_1, \ldots, s_n \rangle)$, where s_i is the sequence played on the i^{th} game.

As above, we define the game for $\forall^2$ by switching the roles of the players on the game for $\exists^2$.

Now we define the game for $\exists^n$ for all $n \in \omega$ in the same way we defined the game for $\exists^2$. We just exchange the $\forall^1$ games for $\forall^{n-1}$ games. Again, the game for $\forall^n$ is the game for $\exists^n$ with the player roles exchanged.

We denote the rules of the game for $\exists^n$ by $rule_n$ and the rules for $\forall^n$ by $\overline{rule_n}$. We use these expressions in game diagrams to represent a game played along these rules.

Lemma 31. *Let $\varphi(x, \alpha)$ be a $(\Sigma^0_2)_{<\omega}$game. Then $(\Sigma^0_2)_{<\omega}$determinacy proves*

$$\exists X_\varphi \forall x((X_\varphi)_x \text{ is a winning strategy for one player in } \varphi(x, \alpha)).$$

Proof. Consider the game where I plays some $x_0 \in \mathbb{N}$ then II decides whether to play $\varphi(x_0, \alpha)$ or $\neg\varphi(x_0, \alpha)$ and whoever wins the game choosen by II wins the whole game. I can not have a winning strategy in this game, so II has a winning strategy by $(\Sigma^0_2)_{<\omega}$determinacy. From this strategy we define X_φ. □

s is winning position of the game for $\exists^n x.\varphi(x)$ iff s is a sequence of games played according to $\overline{rule_{n-1}}$ and I wins $\exists^n x.\varphi(s^\frown x)$. The game for $\exists^n x.\varphi(s^\frown x)$ is also a $(\Sigma^0_2)_{<\omega}$-game, so we can define

$$W^n_\varphi = \{s | (X_\psi)_s \text{ is a winning strategy for } I\}$$

where $\psi = \exists^n x.\varphi(s ^\frown x)$. We consider W^n_φ to be a constant of our language.

We will show that W^n_φ is the least fixed point of $\varphi^{\exists^{n-1}}(x, \vec{y}, X, \vec{Y}) = \varphi(s, \vec{y}, \vec{Y}) \vee \forall^{n-1} s ^\frown \langle x \rangle \in X$. It suffices to show $IGF(\varphi^{\exists^{n-1}}, W^n_\varphi, \preceq, \prec)$ for adequate $\preceq^n_\varphi$ and $\prec^n_\varphi$. We also define $\preceq^n_\varphi$ and $\prec^n_\varphi$ via games. We omit this definition.

Having defined all of this, Heinatsch and Möllerfeld[10] prove by induction:

Theorem 32. *$(\Sigma^0_2)_{<\omega}$determinacy proves $IGF(\varphi^{\exists^{n-1}}, W^n_\varphi, \preceq^n_\varphi, \prec^n_\varphi)$.*

The proof of Theorem 32 is quite long, so we omit it here. We note that the only place where we use determinacy in in is to define W^n_φ using Lemma 31.

Corollary 33. *$(\Sigma^0_2)_{<\omega}$determinacy proves that if $\varphi(x)$ is equivalent to a first-order formula, then so are $\exists^n x.\varphi(x)$ and $\forall^n x.\varphi(x)$. Therefore we have comprehension for all $\mathcal{L}_\partial$-formulas without second-order quantifiers and so we can embed the ∂ame theory into $(\Sigma^0_2)_{<\omega}$determinacy.*

Proof. We have that $\exists^n x.\varphi(x) \leftrightarrow \langle\rangle \in W^n_\varphi$ and $\forall^n x.\varphi(x) \leftrightarrow \langle\rangle \notin W^n_{\neg\varphi}$. □

Thus we can conclude that:

Theorem 34 (Heinatsch, Möllerfeld). *μ-arithmetic is equivalent to $(\Sigma^0_2)_{<\omega}$determinacy over ACA_0 over $\mathcal{L}_2$ sentences.*

Open Question 3. *About Bradfield's transfinite extension of the μ-arithmetic:*

- *Can we formalize the tranfinite μ-arithmetic in $\mathcal{L}_\mu$?*
- *If we can formalize it, is it equivalent to the $\mathbf{\Delta}^0_2$ determinacy over ACA_0?*

Open Question 4. *Can we formalize the weak μ-arithmetic in $\mathcal{L}_2$ and obtain the corresponding results?*

Both of these questions involve the problem of formalizing the transfinite levels of the alternation hierarchy and being unable to use ⅁ame as in [10].

4.3 A finer analysis of Möllerfeld's proof

As stated in Theorem 29 above, Möllerfeld[18] showed that the μ-arithmetic and ⅁ame prove the same $\mathcal{L}_2$ sentences. With this theorem, we show that $(\Sigma^0_2)_{<\omega}$determinacy is equivalent to the μ-arithmetic. So we can use Möllerfeld's result of conservation og Π^1_2-CA$_0$ over $(\Sigma^0_2)_{<\omega}$determinacy.

Furthermore, in his dissertation Möllerfeld defines hierarchies of theories μ_n-arithmetic and ⅁ame$_n$ and also establishes the equivalence of μ_n-arithmetic and ⅁ame$_n$ for all n. We make explicit some steps of the above proof so that we can relate these theories and the determinacy for each level of the difference hierarchy.

We say set variables are μ_0-terms. If $\varphi(\vec{X})$ is a formula of $\mathcal{L}$ without second order quantifiers and $\vec{T}$ is a sequence of μ_m-terms with $m \leq n$ then $\varphi(\vec{T})$ is a μ_{n+1}-formula. If $\varphi(x, X)$ is an X-positive μ_n-formula, then $\mu x X.\varphi$ is a μ_{n+1}-term. The μ_n-arithmetic is the subsystem of μ-arithmetic with axioms for least fixed points only for μ_n-terms.

We define ⅁ame$_n$ to be the subsystem of ⅁ame with axioms for the quantifiers $\exists^m$ restricted to $m \leq n$.

From a (simple) analysis of the proof of theorem 30, it follows that

Lemma 35. *For all $n \geq 1$, μ_n-arithmetic implies Σ^δ_{n-1} determinacy.*

By analysing the winning conditions of the games for $\exists^n$, we have that they can be represented in the following pattern:

$$\begin{array}{l|l|l}
\exists^1 & \Sigma^0_1 & \Sigma^\delta_0 \\
\exists^2 & \Sigma^0_2 & \Sigma^\delta_1 \\
\exists^3 & \Sigma^0_2 \wedge \Pi^0_2 & \Sigma^\delta_2 \\
\exists^4 & (\Sigma^0_2 \wedge \Pi^0_2) \vee \Sigma^0_2 & \Sigma^\delta_2 \vee \Sigma^0_2 \\
\exists^5 & (\Sigma^0_2 \wedge \Pi^0_2) \vee (\Sigma^0_2 \wedge \Pi^0_2) & \Sigma^\delta_2 \vee \Sigma^\delta_2 \\
\exists^6 & (\Sigma^0_2 \wedge \Pi^0_2) \vee (\Sigma^0_2 \wedge \Pi^0_2) \vee \Sigma^0_2 & \Sigma^\delta_2 \vee \Sigma^\delta_2 \vee \Sigma^0_2 \\
\exists^7 & (\Sigma^0_2 \wedge \Pi^0_2) \vee (\Sigma^0_2 \wedge \Pi^0_2) \vee (\Sigma^0_2 \wedge \Pi^0_2) & \Sigma^\delta_2 \vee \Sigma^\delta_2 \vee \Sigma^\delta_2 \\
\vdots & \vdots & \vdots
\end{array}$$

By induction we prove the following lemma:

Lemma 36. *a) For all $n \geq 1$,*

$$\Sigma^\delta_{2n} \vee \Sigma^0_2 = \Sigma^\delta_{2n+1} \vee \Sigma^0_2 = \Sigma^\delta_{2n+1},$$

$$\Pi^\delta_{2n-1} \vee \Sigma^0_2 = \Pi^\delta_{2n} \vee \Sigma^0_2 = \Pi^\delta_{2n}.$$

b) For all $n \geq 3$,

$$\Sigma^\delta_{2n-4} \wedge \Sigma^\delta_{2n-3} = \Sigma^\delta_{2n-2},$$
$$\Pi^\delta_{2n-3} \wedge \Pi^\delta_{2n-2} = \Pi^\delta_{2n-1}.$$

c) For all n,

$$\Sigma^\delta_{2n} \vee \Sigma^\delta_2 = \Sigma^\delta_{2n+2}$$

Using this lemma, we get

Lemma 37. *For all $n \geq 1$, Σ^δ_{n-1} determinacy proves $\Game ame_n$.*

Proof. On the proof of Theorem 32 for n, we only need Σ^δ_{n-1} determinacy for the required instance of Lemma 31. □

In conclusion we get

Theorem 38. *For all $n \geq 1$,*

$$\mu_n\text{-}arithmetic \equiv \Sigma^\delta_{n-1} \text{ } determinacy \equiv \Game ame_n$$

In [10], Heinatsch and Möllerfeld show that:

Theorem 39. Π^1_1-CA_0 *and* $(\Sigma^0_2)_{<\omega}$ *determinacy prove the same* Π^1_2 *sentences.*

This was in turn used in [13] to show:

Theorem 40 (Kołodziejczyk, Michalewski)**.** *The following are equivalent over* Π^1_2*-comprehension:*

- *the complementation theorem for non-deterministic tree automata,*
- *the decidability of the* Π^1_3 *fragment of MSO on the infinite binary tree,*
- *the positional determinacy of parity games, and*
- *the determinacy of* $(\Sigma^0_2)_{<\omega}$ *Gale-Stewart games.*

Open Question 5. *If we can extend the formalization of* μ*-arithmetic to the full case, can we obtain a conservation result parallel to Theorem 39? If the answer for this question is yes, which kind of reverse mathematical results can we obtain?*

Bibliography

[1] P. Blackburn, J. van Benthem, and F. Wolter (eds.), *Handbook of modal logic*, Studies in Logic and Practical Reasoning, vol. 3, Elsevier B. V., Amsterdam, 2007.

[2] J. C. Bradfield, *Simplifying the modal mu-calculus alternation hierarchy*, STACS 98 (Paris, 1998), Lecture Notes in Comput. Sci., vol. 1373, Springer, Berlin, 1998, pp. 39–49.

[3] ———, *The modal mu-calculus alternation hierarchy is strict*, Theoret. Comput. Sci. **195** (1998), no. 2, 133–153. Concurrency theory (Pisa, 1996).

[4] J. Bradfield and C. Stirling, *Modal mu-calculi*, Handbook of modal logic, Stud. Log. Pract. Reason., vol. 3, Elsevier B. V., Amsterdam, 2007, pp. 721–756.

[5] J. C. Bradfield, *Fixpoints, games and the difference hierarchy*, Theor. Inform. Appl. **37** (2003), no. 1, 1–15.

[6] J. Bradfield, J. Duparc, and S. Quickert, *Transfinite extension of the mu-calculus*, Computer science logic, Lecture Notes in Comput. Sci., vol. 3634, Springer, Berlin, 2005, pp. 384–396.

[7] ———, *Fixpoint alternation and the Wadge hierarchy* (2010).

[8] E. Grädel, W. Thomas, and T. Wilke (eds.), *Automata, logics, and infinite games*, Lecture Notes in Computer Science, vol. 2500, Springer-Verlag, Berlin, 2002. A guide to current research.

[9] T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[10] C. Heinatsch and M. Möllerfeld, *The determinacy strength of Π^1_2-comprehension*, Ann. Pure Appl. Logic **161** (2010), no. 12, 1462–1470.

[11] P. G. Hinman, *Recursion-theoretic hierarchies*, Springer-Verlag, Berlin-New York, 1978. Perspectives in Mathematical Logic.

[12] R. Kaye, *Models of Peano arithmetic*, Oxford Logic Guides, vol. 15, The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.

[13] L. A. Kołodziejczyk and H. Michalewski, *How unprovable is Rabin's decidability theorem?*, Proceedings of the 31st Annual ACM-IEEE Symposium on Logic in Computer Science (LICS 2016), ACM, New York, 2016, pp. 10.

[14] D. Kozen, *Results on the propositional μ-calculus*, Theoret. Comput. Sci. **27** (1983), no. 3, 333–354.

[15] W. Li, *Automata-theoretic study on infinite games and fragments of modal μ-calculus*, PhD Thesis (2018).

[16] R. S. Lubarsky, *μ-definable sets of integers*, J. Symbolic Logic **58** (1993), no. 1, 291–313.

[17] M. O. MedSalem and K. Tanaka, *Δ^0_3-determinacy, comprehension and induction*, J. Symbolic Logic **72** (2007), no. 2, 452–462.

[18] M. Möllerfeld, *Generalized Inductive Definitions. The μ-calculus and Π^1_2-Comprehension*, PhD Thesis (2002).

[19] Y. N. Moschovakis, *Descriptive set theory*, 2nd ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.

[20] L. Ong, *Automata, Logic and Games*, Lecture Notes (2015).

[21] S. G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009.

[22] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis. I. Δ^0_2 games*, Z. Math. Logik Grundlag. Math. **36** (1990), no. 6, 481–491.

[23] K. Yoshii, *A survey of determinacy of infinite games in second order arithmetic*, Ann. Japan Assoc. Philos. Sci. **25** (2017), 35–44.